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ENGINEERING MECHANICS

DYNAMICS

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2013

Contents

CHAPTER 1

KINEMATICS OF PARTICLES

Kinematics is the study of motion without regard to the force or forces which influence the motion

1.1) The particle

It is a body neglecting its volume and internal structure i.e. it is considered as concentrated in a geometrical point. We use this concept to simplify the study of the motion.

1.2) General description of a motion of a particle

Consider a particle "P". let "O" be an arbitrary point in the three dimensions space which is considered as the origin. Let "r" be the position vector of "P" with respect to "O". let "t" denotes to the time, if " r " changes with respect to time "t", we say that "P" moves with respect to the origin "O" i.e.

$$\vec{r} = \vec{r}(t)$$

in classical mechanics, the main role is to obtain "$\vec{r}(t)$" for any mechanical system under some certain conditions. The derivative of $\vec{r}$ with respect to time "t" is known as the velocity of point "P" relative to "O".

If is clear that the velocity is a vector and is denoted by $\vec{v}$

$$\vec{v}(t) = \frac{d\,\vec{r}(t)}{dt}$$

if $\vec{v}$ is a constant vector, we say that "P" moves with uniform velocity relative to "O". i.e. the particle moves in a straight line with constant speed. If $|\vec{v}|$ is zero, then "P" is at rest with respect to "O". i.e. the state of rest is a special case of a motion with regular uniform velocity .

$$\text{i.e.} \quad \vec{v} = constant = 0$$

The acceleration is a vector quantity is defined $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

1.3) <u>Rectilinear Motion</u>

Rectilinear motion is motion of a point "P" along a straight line which for convenience here will be chosen as the x-axis. Vector symbols are omitted in this part.

- The position of a point "P" at any time is expressed in times of its distance "x" from a fixed origin "O" on the x-axis. This distance "x" is positive or negative.
- The instantaneous velocity "v" of a point "P" at "t" is $v = \frac{dx}{dt}$
- The instantaneous acceleration "a" of a point "P" at "t" is $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ also $a = \frac{dv}{dx} . \frac{dx}{dt} = v\frac{dv}{dx}$
- For constant acceleration " a=k " the following formula are valid

$$v = v_0 + kt$$

$$v^2 = v_0^2 + 2kx$$

$$x = v_0 t + \tfrac{1}{2}kt^2$$

$$x = \tfrac{1}{2}(v_0 + v)t$$

Where v_0 is initial velocity, v is final velocity, t is time , x is the displacement.

SOLVED EXAPLES

1. a particle moves on a vertical axis with an acceleration **a = 2**$\sqrt{v}$, when **t =2** s its displacement **x = 64/3** ft and its velocity **v=16** ft/sec. determine the displacement, velocity, and acceleration of the particle at **t=3** sec

Solution

Since $a = \frac{dv}{dt}$, then $2\sqrt{v} = \frac{dv}{dt}$ separating the variables, $\frac{dv}{\sqrt{v}} = 2dt$ by integrating

$2\sqrt{v} = 2t + C_1$ but v=16 ft/s when t=2 sec, hence $C_1 = 4$ - the equation becomes

$\sqrt{v} = t + 2$ or $v = (t+2)^2$ .... (1)

since $v = \frac{dx}{dt}$, then $(t+2)^2 = \frac{dx}{dt}$ separating the variables, $dx = (t+2)^2\, dt$ by integrating

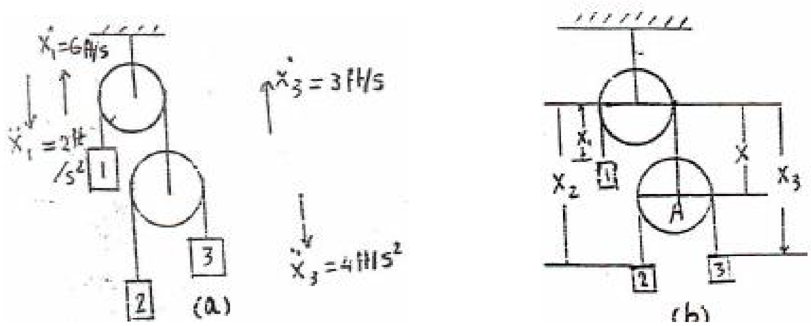
$x = \frac{1}{3}(t+2)^3 + C_2$ but x=64/3 ft when t=2 sec, hence $C_2 = 0$ - the equation becomes

$x = \frac{1}{3}(t+2)^3$ .... (2)

<u>when t=3 sec</u>

$$x = \frac{1}{3}(3+2)^3 = 41.7\ ft\ , \quad v = (3+2)^2 = 25\ ft/s \ \text{ and } \ a = 2\sqrt{v} = 2\sqrt{25} = 10\ ft/s^2$$

2. in the system shown in the figure. Determine the velocity and acceleration of block "2" at the shown instant.



<u>Solution</u>

Fig (b) is drawn to show the position of each weight relative to the fixed line. The length of the cord between the weight (1) and point "A" is constant and equals to one-half the circumference of pulley "B". The length of the cord between weights (2) and (3) is constant and equals to one-half the circumference of pulley "A" plus $x_2 - x + x_3 - x$, thus $x_1 + x = constant$ and $x_2 + x_3 - 2x = constant$

The time derivatives then show

$\dot{x}_1 + \dot{x} = 0$ ………………………. (1)

$\ddot{x}_1 + \ddot{x} = 0$ ……………..……... (2)

$\dot{x}_2 + \dot{x}_3 - 2\dot{x} = 0$ ………...…….. (3)

$\ddot{x}_2 + \ddot{x}_3 - 2\ddot{x} = 0$ ………………… (4)

Calling the upward direction positive and substituting $\dot{x}_1 = 6\ ft/s$ into equation (1) we find $\dot{x} = -6\ ft/s$

Substituting this value together with $\dot{x}_3 = 3\ ft/s$ into equation (3) we find $\dot{x}_2 = -15\ ft/s$

Similar reasoning for the accelerations show $\ddot{x} = 2\ ft/s^2$ and $\ddot{x}_2 = 8\ ft/s^2$

1.4) CURVILINEAR MOTION

Curvilinear motion in a plane is motion along curve (path). The velocity and acceleration of a point will be expressed in

a) Rectangular components
b) Tangential and normal components
c) Radial and transverse components

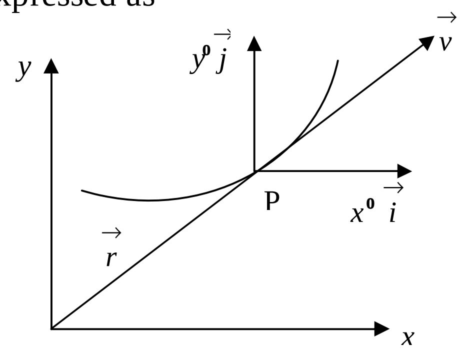
1.4.1) Rectangular components

The position vector $\vec{r}$ of a point "P" on such a curve in term of the unit vectors $\vec{i}$ and $\vec{j}$ along the x and y axes respectively is written

$$\vec{r} = x\,\vec{i} + y\,\vec{j}$$

As point "P" moves, $\vec{r}$ changes and the velocity $\vec{v}$ can be expressed as

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} = x^{\circ}\,\vec{i} + y^{\circ}\,\vec{j}$$



The speed of point is the magnitude of the velocity $\vec{v}$, that is

$$\left|\vec{v}\right| = \sqrt{\left(x^{\circ}\right)^2 + \left(y^{\circ}\right)^2}$$

If q is the angle which the vector $\vec{v}$ makes with the x-axis, we Can write

$$q = tan^{-1}\frac{y^{\circ}}{x^{\circ}} = tan^{-1}\frac{dy/dt}{dx/dt} = tan^{-1}\frac{dy}{dx}$$

Thus, the velocity vector $\vec{v}$ is tangent to the path at point "P"

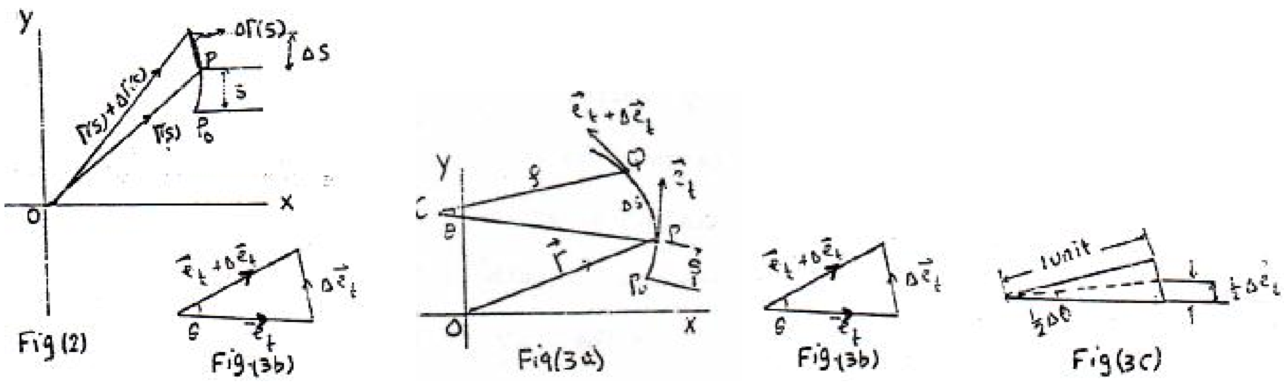
The acceleration vector $\vec{a}$ is the time rate of change of $\vec{v}$, i.e.

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} = x^{\circ\circ}\,\vec{i} + y^{\circ\circ}\,\vec{j}$$

The magnitude of the acceleration $\vec{a}$ is $\left|\vec{a}\right| = \sqrt{\left(x^{\circ\circ}\right)^2 + \left(y^{\circ\circ}\right)^2}$

1.4.2) Tangential and normal components

We now show how to express the vector $\vec{v}$ and $\vec{a}$ in term of the unit vector $\vec{e_t}$ tangent to the path at point "P" and the unit vector $\vec{e_n}$ at right angles to $\vec{e_t}$.



In fig.(2), the point "P" is shown on the curve at a distance "S" along the curve from a reference point "P0". The position vector $\vec{r}$ of point "P" is a function of the scalar quantity "S". To study this relationship, let Q be the point on the curve near "P". The position vectors $\vec{r}(S)$ and $\vec{r}(S)+D\vec{r}(S)$ for point "P" and "Q" respectively are shown as well as the change $D\vec{r}(S)$ which is directed straight line PQ. The distance along the curve from "p" to "Q" is DS. The derivative of $\vec{r}(S)$ with respect to "S" is written

$$\frac{d\vec{r}(S)}{dS} = Lim_{DS\to 0}\frac{\vec{r}(S)+D\vec{r}(S)-\vec{r}(S)}{DS} = Lim_{DS\to 0}\frac{D\vec{r}(S)}{DS}$$

as Q approaches "P", the ratio of the magnitude of the straight line $D\vec{r}(S)$ to the arc length DS approaches unity, also in direction the straight line $D\vec{r}(S)$ approaches the tangent to the path at "P". thus, in the limit, a unit vector $\vec{e_t}$ is defined as follows

$$\frac{d\vec{r}(S)}{dS} = \vec{e_t} \qquad (1)$$

next consider how $\vec{e}_t$ changes with S. As shown in figure (3-a) the center of curvature "C" is a distance r from "P". if we can assume point "Q" close to "P", the unit tangent vectors at "P" and "Q" are $\vec{e}_t$ and $\vec{e}_t + D\vec{e}_t$ respectively.

Since the tangents at "P" and "Q" are perpendicular to the radii drawn to "C", the angle between $\vec{e}_t$ and $\vec{e}_t + D\vec{e}_t$ as shown in figure (3-b) is also Dq. Because $\vec{e}_t$ and $\vec{e}_t + D\vec{e}_t$ are unit vectors, $D\vec{e}_t$ represents only a change in direction (not magnitude). Thus the triangle in figure (3-b) is isosceles and is shown drawn to a large scale in figure (3-c).

It is evident that $\dfrac{\left|\frac{1}{2} D\vec{e}_t\right|}{1} = \sin \frac{1}{2} Dq \cong \frac{1}{2} Dq$ from which $\left|D\vec{e}_t\right| \cong Dq$ but from figure (3-a)

$DS = P\,Dq$ hence we can write $DS \cong r\left|D\vec{e}_t\right|$ thus $\lim_{DS \to 0} \dfrac{\left|D\vec{e}_t\right|}{DS} = \dfrac{1}{r}$

Also, in the limit $D\vec{e}_t$ is perpendicular to $\vec{e}_t$, and is directed toward the center of curvatures, then $\dfrac{d\vec{e}_t}{dS} = \lim_{DS \to 0}\left[\left|D\vec{e}_t\right| / Ds\right]\vec{e}_n = \left(\dfrac{1}{r}\right)\vec{e}_n$ (2)

The velocity vector $\vec{v}$ may now be given in terms of the unit vectors $\vec{e}_n$ and $\vec{e}_t$. Using equation (1)

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dS}\frac{dS}{dt} = S^{o}\,\vec{e}_t \qquad (3)$$

the acceleration vector $\vec{a}$ is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(S^{o}\,\vec{e}_t\right) = S^{oo}\,\vec{e}_t + S^{o}\,\frac{d\vec{e}_t}{dt}$$

But $\dfrac{d\vec{e}_t}{dt} = \dfrac{d\vec{e}_t}{dS}\cdot\dfrac{dS}{dt} = \dfrac{S^{o}}{r}\vec{e}_n$

Then
$$\vec{a} = S^{oo}\,\vec{e}_t + \frac{S^{o2}}{r}\vec{e}_n \qquad (4)$$

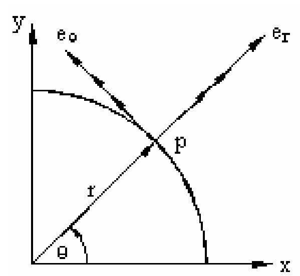
1.4.3) Radial and Transverse components

The velocity vector $\vec{v}$ and acceleration vector $\vec{a}$ are now derived in terms of unit vectors along and perpendicular to the radius vectors.

Note that there is an infinite set of unit vectors because any point my be chosen as a pole.

The radius vector $\vec{r}$ makes an angle f with x-axis. The unit vector $\vec{e}_r$ is chosen outward along $\vec{r}$. The unit vector $\vec{e}_f$ is perpendicular to $\vec{r}$ and in the direction of increasing f

i.e. $$\vec{r} = r\,\vec{e}_r \qquad (5)$$



The velocity vector $\vec{v}$ is $\vec{v} = \vec{r^{o}} = r^{o}\,\vec{e}_r + r\,\vec{e^{o}}_r$ where $\overset{o}{\vec{e}}_r = \dfrac{d\vec{e}_r}{dt}$

to evaluate $\overset{o}{\vec{e}}_r$ and $\overset{o}{\vec{e}}_f$, allow "P" to move to a nearly point Q with a corresponding set of unit vectors $\vec{e}_r + D\vec{e}_r$ and $\vec{e}_f + D\vec{e}_f$ as shown in figure (5) the limit $d\vec{e}_r$ has a magnitude df in the $\vec{e}_f$.

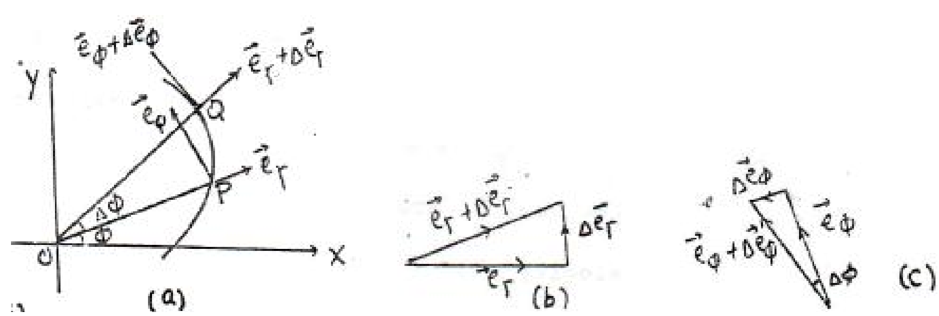


figure (5)

$$\overset{o}{\vec{e}}_r = \frac{d\vec{e}_r}{df}\cdot\frac{df}{dt} = f^{o}\,\vec{e}_f \qquad \text{and} \qquad \overset{o}{\vec{e}}_f = \frac{d\vec{e}_f}{df}\cdot\frac{df}{dt} = -f^{o}\,\vec{e}_f$$

where f^{o} is the time rate of the angle f which the radius vector $\vec{r}$ makes with the x-axis.

The velocity vector $\vec{v}$ may now be written as $$\vec{v} = \vec{r^{0}} = r^{0}\,\vec{e}\,r + r f^{0}\,\vec{e}\,f \qquad (6)$$

The acceleration $\vec{a}$ is the time derivative of the terms in equation (6)

$$\vec{a} = \ddot{r}\,\vec{e}_r + \dot{r}\,\dot{\vec{e}}_r + \dot{r}\,\dot{f}\,\vec{e}_f + r\,\ddot{f}\,\vec{e}_f + r\,\dot{f}\,\dot{\vec{e}}_f$$

$$\vec{a} = \ddot{r}\,\vec{e}_r + \dot{r}\,\dot{f}\,\vec{e}_f + \dot{r}\,\dot{f}\,\vec{e}_f + r\,\ddot{f}\,\vec{e}_f - r\,\dot{f}^2\,\vec{e}_r$$

where $\ddot{f}$ is the angular acceleration (time derivative of the angular velocity $\dot{f}$) i.e.

$$\vec{a} = \left(\ddot{r} - r\,\dot{f}^2\right)\vec{e}_r + \left(2\dot{r}\,\dot{f} + r\,\ddot{f}\right)\vec{e}_f \qquad (7)$$

as a special case of curvilinear motion consider a point moving in a circular path of radius "R" , substituting "R" for "r" in equations (6) and (7), noting $\dot{R} = \ddot{R} = 0$, We obtain

$$\vec{v} = R\,\dot{f}\,\vec{e}_f \quad \text{and} \quad \vec{a} = \left(-R\,\dot{f}^2\right)\vec{e}_r + \left(R\,\ddot{f}\right)\vec{e}_f \qquad (8)$$

Thus the acceleration has a tangent component of magnitude $\left(R\,\ddot{f}\right)$ and a normal component directed toward the center of magnitude $R\,\dot{f}^2$.

SOLVED EXAPLES

3. show that the curvature of a plane curve at point "P" may be expressed as

$$\frac{1}{r} = \frac{x' y'' - x'' y'}{\left(x'^{2} + y'^{2}\right)^{3/2}}$$

Solution

From the calculus, the curvature of any curve y=f(x) at a point "P" is

$$\frac{1}{r} = \frac{\left(d^2 y / dx^2\right)}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}} \quad \ldots\ldots. (1)$$

but $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{y'}{x'}$ and $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\left(\frac{dt}{dx}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\left(\frac{1}{x'}\right) = \frac{x' y'' - x'' y'}{x'^{3}}$

substituting into (1), we obtain the required equation.

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4. particle describe the path y=4x² with constant speed v, where x and y are in meters. What is the normal component of the acceleration ?

Solution

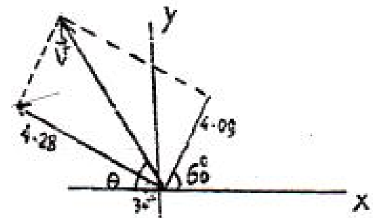
$$\text{since } \frac{1}{r} = \frac{\left(d^2 y / dx^2\right)}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}} = \frac{8}{\left(1 + (8x)^2\right)^{3/2}}$$

$$a_n = \frac{v^2}{r} == \frac{8v^2}{\left(1 + (8x)^2\right)^{3/2}} \text{ m/s}$$

5. a particle moves on curve of equation of path **r=2θ** if the angle **θ = t²** , determine the velocity and the acceleration of the particle when **θ = 60°** use two methods.

Solution

a- Polar Coordinates



since $q = t^2$, $q^{\mathbf{o}} = 2t$, $r = 2q = 2t^2$, $r^{\mathbf{o}} = 4t$

The velocity v at $q = \frac{p}{3}$ radians is formed as follow

$$q = \frac{p}{3} = t^2 \quad or \quad t = 1.033\,sec$$

$$\vec{v} = r^{\mathbf{0}}\,\vec{e}_r + r\,f^{\mathbf{0}}\,\vec{e}_f = 4(1.033)\,\vec{e}_r + [(2*1.047)(2*1.033)]\,\vec{e}_f$$

$$\vec{v} = 4.09\,\vec{e}_r + 4.28\,\vec{e}_f$$

And $v = \sqrt{(4.09)^2 + (4.28)^2} = 5.92\ ft/s$ with $q_x = 30^{\mathbf{o}} + tan^{-1}\left(\frac{4.09}{4.28}\right) = 73.7^{\mathbf{o}}$

The acceleration $\vec{a} = \left(r^{\mathbf{00}} - r\,f^{\mathbf{0}2}\right)\vec{e}r + \left(2r^{\mathbf{0}}f^{\mathbf{0}} + r\,f^{\mathbf{00}}\right)\vec{e}f$

$$\vec{a} = -4.77\,\vec{e}r + 20.94\,\vec{e}f$$

And $a = \sqrt{(-4.77)^2 + (20.94)^2} = 21.5\ ft/s^2$ with $q_x = 30^{\mathbf{o}} - tan^{-1}\left(\frac{-4.77}{20.94}\right) = 17.2^{\mathbf{o}}$

b- Cartesian Coordinates

$x = r\,cosq = 2q\,cosq = 2t^2\,cos\,t^2$ and $y = r\,sinq = 2q\,sinq = 2t^2\,sin\,t^2$

then $x^{\mathbf{o}} = 4t\,cos\,t^2 - 4t^3\,sin\,t^2 = -1.66\ ft/s$

$y^{\mathbf{o}} = 4t\,\mathbf{sin}\,t^2 + 4t^3\,\mathbf{cos}\,t^2 = 5.68\ ft/s$

hence $v = \sqrt{(-1.66)^2 + (5.86)^2} = 5.92\ ft/s$ with $q\ = tan^{-1}\left(\frac{5.58}{1.66}\right) = 73.7^{\mathbf{o}}$

The acceleration

then $x^{oo} = 4\cos t^2 + 4t\left(-2t\sin t^2\right) - 12t^2\sin t^2 - 8t^4\cos t^2 = -20.52\ ft/s^2$

$y^{oo} = 4\sin t^2 + 8t\cos t^2 + 12t^2\cos t^2 - 8t^4\sin t^2 = 6.34\ ft/s^2$

hence $a = \sqrt{(-20.52)^2 + (6.34)^2} = 21.5\ ft/s^2$ with $q = \tan^{-1}\left(\frac{6.34}{20.52}\right) = 17.2^{o}$

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6. A fly wheel 1.2 m in diameter accelerates uniformly from rest , to **2000** r.p.m, in **20** sec. what is the angular acceleration?

Solution

Since constant acceleration is involved then the formulas of constant acceleration may be used.

i.e. the angular motion is similar to rectilinear motion, i.e. "ω" replaces "v", "θ" replaces "x"and "α" replaces "a".

The wheel starts from rest , hence ω_0=zero.

The formula involving these four symbols is $\boldsymbol{\omega = \omega_0 + \alpha\, t}$

ω = 2 Л N/60 = 2 Л 2000 /60 = **209** rad/sec

hence

$$\boldsymbol{\alpha = \omega - \omega_0 / t = 209/20 = 10.5\ rad/s^2}$$

PROBLEMS

1) A particle begins its motion in a straight line such that its position relative to a fixed point on that straight line is given by : $X = t^3 - 9t^2 + 15t + 5$ Find the position of the particle when the acceleration vanished, Find also the total distance.

2) A particle moves in a straight line with acceleration $a = -5$ cm / sec² when $t = 0$, the particle is at the origin and its velocity $v_0 = 20$ cm / sec. Find the velocity and the distance covered when $t = 6$ sec.

3) The acceleration of a particle moves in a straight line is $a = 5 - 2V$ where V is its velocity if the particle begins its motion from rest, Find the time needed to the velocity becomes 1.25 m / sec.

4) A particle moves in a straight line with acceleration $a = \mathbf{m}\, t^2$. Where **m** is constant find the value of **m** if (v = -32 m/s at t = 0 sec) and (v = 32 m/s at t = 4 sec). Also find the position (x) at any instant if the particle released from rest.

5) A particle moves in a straight line such that v = 2 / 1+x where v is the velocity Find the time needed for particle to arrive the point x = 4 cm. Find also its acceleration. If the particle begins its motion from rest from the origin.

6) Find the velocity and the acceleration at t=1 sec of a particle moving with

$$x = (t+1)^2 \quad \text{and} \quad y = (t+1)^{-2}$$

7) The plane motion of a particle is defined by the equations $r = t^3 - 3t^2$ and $q = t^3 - 4t$ find the velocity and the acceleration at t=1 sec.

8) find the radial and the transverse components of a moving particle such that

$r = k(1 + \sin t)$ *and* $f = 1 - e^{-t}$ where k is constant and t is the time.

9) A particle moves on the curve $r = k(1 - \sin t)$ find the radial and the transverse components of the acceleration knowing that the particle moves with constant angular velocity **3** rad/s.

10) If the radial and transverse components of the velocity of a particle are respectively ar^2, $2ar^2$, where a is a constant. Find the components of the acceleration, find also the equation of the path, knowing that $r = a$ when $\theta = 0$.

11) A fly wheel **1.2** m in diameter accelerates uniformly from rest , to **2000** r.p.m, in **20** sec. what is the angular acceleration? also determine the linear velocity and linear acceleration of a point on the rim of the fly wheel .

12) An automobile travels at a constant speed on a highway curve of **1000** m radius. if the normal component of the acceleration is not to exceed **1.2** m/s^2, determine the maximum velocity.

13) A car begins motion from rest with uniform tangential acceleration **3** ft/sec^2 on a curved road with radius of curvature **400** ft. Find the covered distance before the acceleration reaches **6** ft / sec^2.

14) find the tangential and normal components of the acceleration of a particle moving by the following equations. $X = 2t + 3$ & $Y = t^2 - 1$

15) Prove that the acceleration of a point moving in a curve with uniform speed is $\rho\dot{\Psi}^2$. Such that ρ is the curvature and Ψ is the inclination angle.

CHAPTER 2

MOTION OF A PROJECTIES

The object of the science of dynamics is to investigate the motion of bodies as affected by the forces which act upon them.

2.1) Newton's laws of motion

1-A particle will maintain its state of rest or of uniform motion (constant speed) along a straight line unless compelled by some force to change that state. In other words, a particle accelerates only if an unbalanced force acts on it.

2- The time rate of change of the product mass and velocity of a particle is proportion to the force acting on the particle. The product of the mass "m" and the velocity "v" is the linear momentum "G". thus the second law states :

$$\vec{F} = k\frac{d\left(m\vec{v}\right)}{dt} = k\frac{d\vec{G}}{dt}$$

If "m" is constant the above equation becomes

$$\vec{F} = km\frac{d\vec{v}}{dt} = km\vec{a}$$

If a suitable units are chosen so that constant of proportionality $k = 1$, these equations are

$$\vec{F} = \frac{d\vec{G}}{dt} \quad or \quad \vec{F} = m\vec{a}$$

3- to every action of force, there is an equal and opposite reaction, or force. In other word, if a particle exerts a force on a second particle, then the second particle exerts a numerically equal and opposite directed force on the first particle.

2.2) Acceleration

Acceleration of a particle may be determined by the vector equation

$$\Sigma \vec{F} = m\vec{a} = m\vec{\ddot{r}}$$

Where $\Sigma \vec{F} =$ vectors sum of all the forces acting on the particle, "m" is the mass of the particle, $\vec{a} = \vec{\ddot{r}} =$ acceleration.

2.3) D'alembert's principal

jean D'alembert in 1743 suggested that Newton's second law of motion could be written

$$\Sigma \vec{F} - m\vec{a} = 0$$

Thus an imaginary force called an "inertia force", which is collinear with the $\Sigma \vec{F}$ but oppositely sensed and magnitude $m\vec{a}$ would if applied to the particle cause it to be in equilibrium. the equations of equilibrium would then apply actually the particle is not in equilibrium, but the equations of motion can be applied in the form shown above.

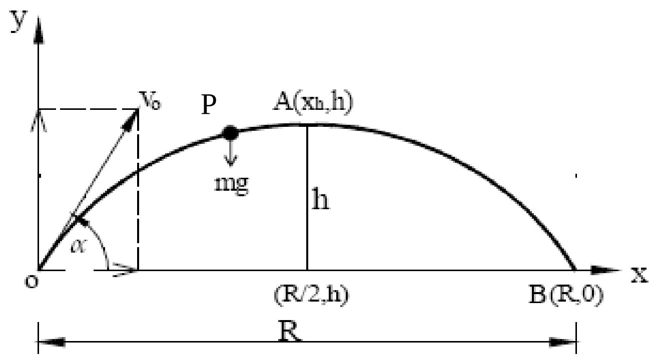
2.4) Problems in dynamics

The solutions of problems in dynamics vary with the type of force system. Many problems involve forces which are constant. In other problems the force vary with the distance.
The subject of ballistics is introduce in an elementary manner, which deals with the motion of a projectile under the action of the constant force of gravity.

2.5) The Projectiles

Consider the gravity as the only one acting on the projectile i.e. neglect the resistance of the air, the rotation of the earth about itself and about the sun. This is due to the short time needed to the motion of the projectile. We also neglect the effect of the configuration of the projectile on the motion i.e. it is consider as a particle.

To study the motion of a particle (projectile), let $\vec{v}_0$ be the initial velocity in a direction makes an angle a with the horizontal. Consider the orthogonal Cartesian axes x-y in the plane of motion such that the origin is coinciding with the point of projection. Let p(x,y) be any point of the path of the projectile, and $\vec{r}$ its position vector i.e.



$\vec{r} = x\,\vec{i} + y\,\vec{j}$

$\vec{r}^{\,o} = x^{o}\,\vec{i} + y^{o}\,\vec{j}$

$\vec{r}^{\,oo} = x^{oo}\,\vec{i} + y^{oo}\,\vec{j}$

The equation of motion is $m\vec{r}^{\,oo} = -mg\,\vec{j}$ where m is the mass of the particle, then $\vec{r}^{\,oo} = -g\,\vec{j}$

this equation can be written in terms of its components as

$x^{oo} = 0$ ……………………………………………………………………………………(1)

$y^{oo} = -g$ ……………………………………………………………………………………(2)

integrate (1),(2) with respect to the time "t", we get the velocity components

$$x^{o} = C_1 \quad \text{and} \quad y^{oo} = -gt + C_2$$

assuming that at $t = 0,\ x^{o} = (v_0 \cos a),\ y^{o} = (v_0 \sin a)$ therefore $C_1 = (v_0 \cos a),\ C_2 = (v_0 \sin a)$

$x^{o} = v_0 \cos a$ ……………………………………………………………………………(3)

$y^{o} = -gt + v_0 \sin a$ ……………………………………………………………………(4)

integrate (3),(4) with respect to the time "t", we get the position of the projectile at any instant

$x = (v_0 \cos a)t + C_3$

$y = -\frac{1}{2} g t^2 + (v_0 \sin a)t + C_4$

since the projectile fired from the origin (x=0,y=0) at t=0, then $C_3 = C_4 = 0$

$x = (v_0 \cos a)t$ ……………………………………………………………..……………..(5)

$y = -\frac{1}{2} g t^2 + (v_0 \sin a)t$ …………………………………………………………...……….(6)

The equation of the path as a relation between *x* and *y* may be obtained by eliminate "t" between (5) and (6) i.e.

$y = x \tan a - \frac{g x^2}{2 v_0^2 \cos^2 a}$ ……………………………………………………………………..(7)

which is the equation of a parabola with vertex upward and a vertical axis.

Discussion of the motion

The above motion may be decomposed into horizontal and vertical motions. The horizontal motion is according to a uniform velocity $v_0 \cos a$. This is because the weight of the body works only in the vertical direction.

Whether the vertical motion is similar to the motion of a projectile under the effect of a constant gravity.

2.5.1) Maximum Height

when the projectile arrives to the highest point "A" in its path, the vertical component of the velocity vanishe, and equation (4) becomes

$$0 = -g t_h + v_0 \sin a$$

where t_h is the time needed to arrive "A" then

$t_h = \frac{v_0 \sin a}{g}$ ………………………………………………………………………………(8)

The maximum height is obtained when $t_h = t$ i.e. from equation (6)

$$h = \frac{\left(v_0^2 \sin^2 a\right)}{2g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

the x-component of "A" is obtained from (5) and (8) i.e.

$$x_h = \frac{\left(v_0^2 \sin 2a\right)}{2g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

the last result may be obtained from the Cartesian equation of the path (7) by remarking that "A" is the maximum point of the path.

2.5.2) The range on a horizontal plane

it is the distance between the point of projection "O" and the point of intersection of the path and the horizontal plane which passes through "O" .

Since the y-coordinates of the point "B" is equal to zero. Then, equation (6) gives

$$0 = -\tfrac{1}{2} g t^2 + \left(v_0 \sin a\right) t$$

the root (t=0) gives the origin "O", while the second root gives the point "B"

$$t_R = \frac{2 v_0 \sin a}{g} = 2 t_h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

is called the time of flight, which is the time needed for the projectile to arrive "B" from "O". it is equal twice the time of maximum height. To get "R" put $t_R = t$ into equation (5), we get

$$R = \frac{\left(v_0^2 \sin 2a\right)}{g} = 2 x_h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

i.e. the path of the projectile is similar about the vertical line passes through the point "A".

equation (12) shows that the angle of projection are a and $\frac{p}{2} - a$.

2.5.3) Maximum Range

the maximum range is obtained when "R" is a maximum, i.e. when $\sin 2a = 1$. i.e. $a = \frac{p}{4}$

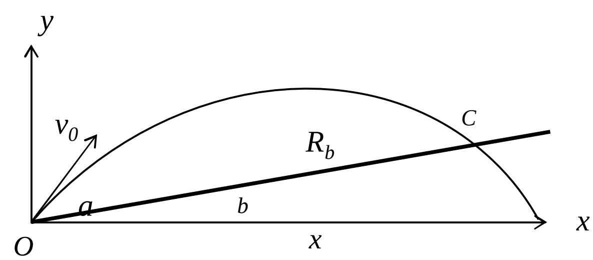
$$R_{max} = \frac{v_0^2}{g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

the angle of projection is $a = \frac{p}{4}$

2.5.4) The range on an inclined plane

let the particle be projected from "O" upward plane passes through "O" and inclines with angle b on the horizontal. Let v_0 makes an angle a with OX such that $a > b$. To find R_b (the length of OC), get the coordinates of which is the point of intersection of the path

$y = x \tan a - \dfrac{g x^2}{2v_0^2 \cos^2 a}$ with the plane $y = x \tan b$



i.e. $x \tan b = x \tan a - \dfrac{g x^2}{2v_0^2 \cos^2 a}$

i.e. $\dfrac{g x}{2v_0^2 \cos^2 a} = \dfrac{\sin a}{\cos a} - \dfrac{\sin b}{\cos b}$

i.e. $\dfrac{g x}{2v_0^2 \cos^2 a} = \dfrac{\sin(a - b)}{\cos a \ \cos b}$

i.e. $x_c = \dfrac{2v_0^2 \cos a \ \sin(a - b)}{g \cos b}$ then the range $R_b = \dfrac{x}{\cos b}$

$$R_b = \frac{2v_0^2 \cos a \ \sin(a - b)}{g \cos^2 b} \quad \ldots\ldots(15)$$

to get the time of flight T_b on the inclined plane, put x=xc into equation (5)

$(v_0 \cos a) T_b = \dfrac{2v_0^2 \cos a \ \sin(a - b)}{g \cos b}$ i.e.

$$T_b = \frac{2v_0 \sin(a - b)}{g \cos b} \quad \ldots\ldots(16)$$

to get the angles of projection corresponding to the maximum range, put equation (15) into the form

$$R_b = \frac{v_0^2 [\sin(2a - b) - \sin b]}{g \cos^2 b}$$

we find that the angles of projection are $a \, , \ \dfrac{p}{2} - a + b$ ……(17)

the last equation shows that the same range is obtained when the angles of projection are a_1, a_2 where a_1 and a_2 make the same angle q with the line bisecting the angle between the inclined plane and the vertical upward assuming the same velocity of projection.

$a_{1,2} = \frac{p}{4} + \frac{b}{2} \pm q$ where q is any angle.

The maximum range on the inclined plane R_{bm} is obtained from equation (15), by putting

$sin(2a - b) = 1$ i.e. $(2a - b) = \frac{p}{2} \Rightarrow a_m = \frac{p}{4} + \frac{b}{2}$ ……………………..……………….(18)

Which is the angle gives the maximum range substituting into equation (15) we get the maximum range i.e. $R_{bm} = \frac{v_0^2}{g(1 + sin\, b)}$ ………………………………………………….(19)

2.5.5) The Safety Curve

if we imagine a gun is fixed and project its projectiles with velocity v_0 at different angles. We find two angles to hit any point in the plane of motion. This is clear from the equation of the path

$$y = x\, tan\, a - \frac{g\, x^2}{2v_0^2\, cos^2 a} \quad \text{or} \quad y = x\, tan\, a - \frac{g\, x^2}{2v_0^2}\left(1 + tan^2 a\right)$$

i.e. $\mathbf{tan}^2 a - \frac{2v_0^2}{g\, x}\mathbf{tan}\, a + \frac{2v_0^2 y}{g\, x^2} + 1 = 0$ ………………………………………………………(20)

which is a second degree equation in $tan\, a$ i.e. for every point P(x,y), there are two projected angles

$\mathbf{tan}\, a = \frac{v_0}{g\, x} \pm \sqrt{\frac{v_0^2}{g^2 x^2} - \frac{2v_0^2 y}{g\, x^2} - 1}$ …………………………………………………………(21)

and can be hit the point "P" with these two values of a . This is true only if the discriminate is positive i.e. $y < \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$ ……………………………………………..…………(22)

-if the discriminate is negative then the two angles of projection are imaginary and then we cannot hit the point. i.e. $y > \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$ ………………………………………………..(23)

if the discriminate is zero then there is only one angle of projection, which hit the point "O" i.e. $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$ ……………………………………………………………………(24)

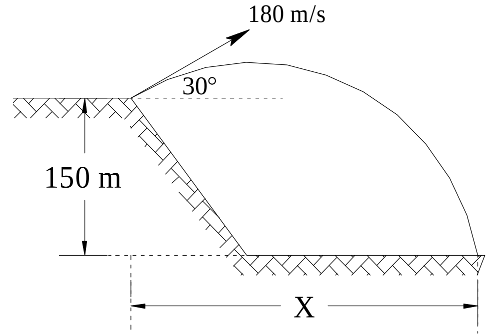
The curve (24) is called the safety curve because it separates between two regions, the first one is determined by equation (22) and any point in this region can be hitted, while the second one is determined by (23) and any point cannot be at all. This region is called a safety region.

SOLVED EXAMPLES

1-A projectile is fired from the edge of a **150** m cliff with an initial velocity of **180** m/s , at an angle of **30**° with the horizontal. Find

a) the horizontal distance from the gun to the point where the projectile strikes the ground.

b) the greatest elevation above the ground.



Solution

Given the initial velocity $v_0 = 180 m/s$, $a = 30^{\circ}$ and $y = 150 m$

(a) Since $y = -\frac{1}{2} g t^2 + (v_0 \sin a)t \quad \Rightarrow \quad -150 = -\frac{1}{2}(9.8)t^2 + (180 \sin 30^{\circ})t$

$$\therefore t^2 - 18.36\,t - 30.6 = 0 \quad \Rightarrow t = 19.9\,s \quad or \quad t = -1.5\,s$$

Hence $x = (v_0 \cos a)t = (180 \cos 30^{\circ})(19.9) = 3100\,m$

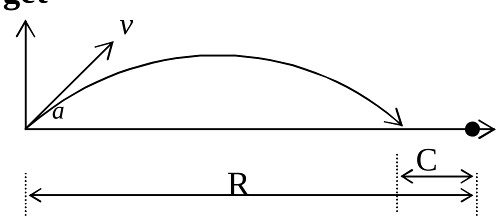
(b) the total height = $150 + \frac{(v_0^2 \sin^2 a)}{2g} = 150 + \frac{(180^2 \sin^2 30^{\circ})}{2(9.8)} = 563.3\,m$

2-A Projectile aimed at a mark which is in a horizontal plane through the point of projection. The projectile fall at (**C**) meter short of the aim when the angle of projection be **α** and goes (**d**) meter too far when the angle of projection be **β**, show that if the velocity of projection be the same in all cases the correct angle of project should be

$$q = \sin^{-1}\left(\frac{C \sin 2b + d \sin 2a}{C + d}\right)$$

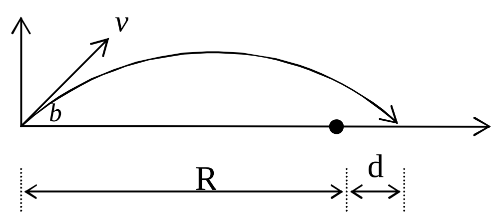
Solution

Let q be the correct angle of projection to hit the target



Case (1) $R - C = \dfrac{v^2 \sin 2a}{g}$ ***(1)***

Case (2) $R + d = \dfrac{v^2 \sin 2b}{g}$ ***(2)***



Case (3) $R = \dfrac{v^2 \sin 2q}{g}$ ***(3)***

Multiply (1) by "d" and (2) by "C" we get

$Rd - Cd = \dfrac{v^2 d \sin 2a}{g}$ **(I)** **and** $RC + Cd = \dfrac{v^2 C \sin 2b}{g}$ **(II)**

Adding (I) and (II) we get

$R(C + d) = \dfrac{v^2}{g}(d \sin 2a + C \sin 2b)$ **or** $R = \dfrac{v^2}{g}\dfrac{(d \sin 2a + C \sin 2b)}{(C + d)}$ **(4)**

by equating both (4) and (3) we get

$\dfrac{v^2 \sin 2q}{g} = \dfrac{v^2}{g}\dfrac{(d \sin 2a + C \sin 2b)}{(C + d)}$ **or** $\sin 2q = \dfrac{(d \sin 2a + C \sin 2b)}{(C + d)}$

$$\therefore q = \frac{1}{2}\sin^{-1}\left[\frac{(d \sin 2a + C \sin 2b)}{(C + d)}\right]$$

3- Two particles are simultaneously projected in the same vertical plane from the same point with velocity (***u*** and ***v***) at angles (**α**, **β**) with the horizontal show that the line joining them moves always parallel to itself and the time that elapses when their velocities are parallel is $t = \frac{uv\sin(a-b)}{g(v\cos b - u\cos a)}$

Solution

(a) let after time "t" the two particles reach A and B, having coordinates (x_1,y_1) and (x_2,y_2)

for the particle reaching **A**,

$y_1 = (u\sin a)t - \frac{1}{2}gt^2$ **and** $x_1 = (u\cos a)t$

for the particle reaching **B**,

$y_2 = (v\sin b)t - \frac{1}{2}gt^2$ **and** $x_2 = (v\cos b)t$

Therefore the slope of AB = $\frac{y_2 - y_1}{x_2 - x_1}$ = $\frac{(v\sin b)t - \frac{1}{2}gt^2 - (u\sin a)t + \frac{1}{2}gt^2}{(v\cos b)t - (u\cos a)t} = \frac{v\sin b - u\sin a}{v\cos b - u\cos a} = \textbf{constant}$

Hence the line joining the two particles will always have the same slope, i.e. the line joining them moves always parallel to it self.

(**b**) now consider the particle "A" suppose its coordinates at any time "t" be (x_1,y_1), therefore the slope of the direction of motion at any time is given by $\frac{y^{\mathbf{0}}}{x^{\mathbf{0}}} = \frac{u\sin a - gt}{u\cos a}$

similarity, the slope of the direction of motion of particle "B" is given by $\frac{y^{\mathbf{0}}}{x^{\mathbf{0}}} = \frac{v\sin b - gt}{v\cos b}$

hence, if at time "t", the direction of motion of the two particles should be the same, we have

$$\frac{u\sin a - gt}{u\cos a} = \frac{v\sin b - gt}{v\cos b}$$

$$(u\sin a - gt)\ \ v\cos b = (v\sin b - gt)\ \ u\cos a$$

$$(v\cos b - u\cos a)gt = uv(\sin a\ \cos b - \sin b\ \cos a)$$

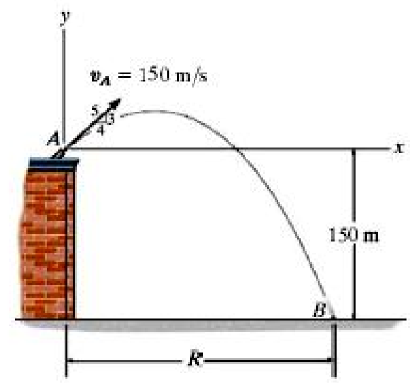
$$(v\cos b - u\cos a)gt = uv\sin(a-b)$$

$$\therefore t = \frac{uv\sin(a-b)}{g(v\cos b - u\cos a)}$$

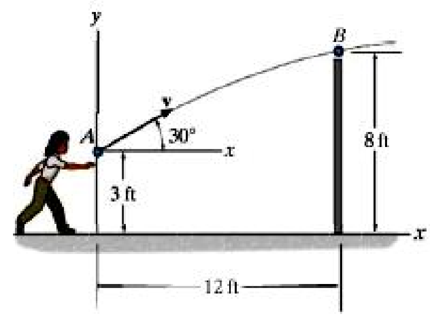
PROBLEMS

1- A projectile is fired with velocity **400** m/s in a direction makes an angle of **30**° with the horizontal. Find a) the time of flight and the range b) the maximum height

2- A projectile is fired with an initial velocity of **150** m/s off the roof of the building. Determine the range **R** where it strikes the ground at **B**.



3- A ball is thrown from **A**. if it is required to clear the wall at **B**. determine the minimum magnitude of its initial velocity.



4- A particle is projected from a certain point . It is noticed that its range on the horizontal plane which passes through the point of projection is equal to **three** times the maximum height above the point of projection, and the velocity after two seconds from the time of projection is equal to the velocity of projection Find the velocity of projection, also find the position of the projectile after **5** sec from the beginning of projection.

5-A ship moves with velocity (v) and carry a gun which project its projectiles in the opposite direction of the motion of the ship. If the gun inclines with an angle (α) to the horizontal. And the initial velocity of the projectile is (v_0) prove that the range is $\frac{2v_0^2 \sin a\left(v_0 \cos a - v\right)}{g}$ and the angle of inclination corresponding to the maximum range is

$$\cos^{-1}\left[\frac{v+\sqrt{v^2+8v_o^2}}{4v_o}\right]$$

6- Two particles **A**, **B** are projected simultaneously from the same point and same velocity (**v**).the motion is in the same vertical plane. Prove that if the two particles are in the of motion, the straight line joining them moves parallel to itself, and makes a fixed angle with the vertical with amount **$(\alpha' + \alpha) / 2$** where **α**, **α'** are the angles of projection with the horizontal. Prove also that the distance between the two particles increases with a constant rate and the velocities of the two particles become parallel after time

$$(v \cos (\alpha' - \alpha) / 2) / g \ \sin (\alpha' + \alpha) / 2) .$$

CHAPTER 3

SIMPLE HARMONIC MOTION

3.1) introduction

The simple harmonic motion is a rectilinear motion in which the acceleration is negatively proportional to the displacement.

If "***k***" denotes the force at unit distance, the force at distance ***x*** will be –***kx***, the sign being always opposite to that of "***x***". the equation of motion is accordingly

$$m\frac{d^2x}{dt^2} = -kx \quad \ldots\ldots(1)$$

if we write

$$w^2 = \frac{k}{m} \quad \ldots\ldots(2)$$

i.e. equation (1) becomes

$$\frac{d^2x}{dt^2} = -w^2 x \quad \ldots\ldots(3)$$

to integrate the equation (3) we write

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) \quad \ldots\ldots(4)$$

Substitute into (3) we get $\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -w^2 x$ integrate both sides, and using the initial conditions $\boldsymbol{at\ \ t = 0\ \ x = x_0, v = v_0}$ ……(5)

We get $v^2 - v_0^2 = -w^2\left(x^2 - x_0^2\right)$ ……(6)

equation (6) shows that the maximum value of the velocity is obtained when the particle is at the center of the motion.

3.2) The velocity

$$v = \frac{dx}{dt} = \pm\sqrt{v_0^2 + w^2 x_0^2 - w^2 x^2} \quad \text{or} \quad v = \pm w\sqrt{\frac{v_0^2}{w^2} + x_0^2 - x^2} \quad \dots\dots(7)$$

equation (7) shows that the speed at any instant depends only on the distance from the center of attraction. Also, we deduce from equation (7) that the motion is possible only on that part of the straight line $x^2 \le x_0^2 + \frac{v_0^2}{w^2}$ or $-\sqrt{x_0^2 + \frac{v_0^2}{w^2}} \le x \le +\sqrt{x_0^2 + \frac{v_0^2}{w^2}}$ ……………….….(8)

3.3) The Amplitude

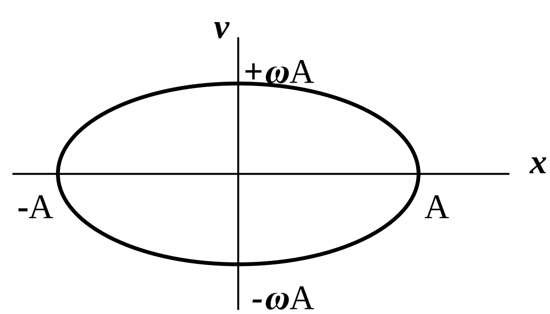
The amplitude is defined as $A = \sqrt{x_0^2 + \frac{v_0^2}{w^2}}$ ……………………………………...…….(9)

Clearly the amplitude of the motion depends only on the initial distance form the center of the attraction, and the value of the initial speed. The speed v vanishes at the points C_1, C_2 i.e. it vanishes at $x_c = -A$, $x_c = +A$.

The maximum value of the velocity is at $x = 0$ i.e. at the center of attraction, where the following relation is satisfied $v_{max} = \pm w A$ ……………………………..……………..….(10)

The figure shows how the velocity changes with the position while the acceleration vanishes at the center attraction, and reaches its maximum value at C_1, C_2.

The maximum value of acceleration is $a_{max} = \pm w^2 A$ ……………………………………(11)



3.4) Description of motion

if the particle is at C_2 then the motion begins from rest and the velocity increases towards the center of attraction, until it reaches its maximum value at the center "O". the value of velocity decrease after then under the influence of the attraction force until the particle be at rest at C_1. A similar motion begins from C_1 to C_2 , etc.

to calculate the relation between the distance and the time, rewrite equation (7) in the form

$$\frac{dx}{dt} = \pm w\sqrt{A^2 - x^2}$$

integrate both sides with respect to t

$$\pm \int \frac{dx}{\sqrt{A^2 - x^2}} = w\,t + \boldsymbol{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

the constant of integration is determined from the initial condition of the position of the particle. The student can easily prove that any of the two signs gives the same result of the integration i.e.

$$\cos^{-1}\left(\frac{x}{A}\right) = w\,t + g \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Where g is a new constant. Then

$$x = A\cos\left(w\,t + g\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

The angle $\left(w\,t + g\right)$ is called the phase angle, and g is the initial phase angle. The relation between the velocity and the time is obtained from (13)

$$v = \frac{dx}{dt} = -wA\sin\left(w\,t + g\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

the initial value of the phase g is obtained when $t = 0$ i.e.

$$\cos g = \frac{x_0}{A} \text{ and } \sin g = -\frac{v_0}{w\,A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

3.5) the periodic time

The time that elapses from any instant until the instant in which the moving point is again moving through the same position with the same velocity and direction is called the periodic time of the motion.

It is clear that the periodic time correspond the increase of phase angle $(w t + g)$ with amount $2p$.

If t_1 and t_2 are the instants of time at which the particle exist at C_1 two consecutive times, then $(w t_2 + g) = (w t_1 + g) + 2p$.

$$t = t_2 - t_1 = \frac{2p}{w} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

i.e. t is independent of the position of C_1.

3.6) the frequency

the frequency N is the inverse of the periodic time $N = \frac{w}{2p}$ i.e. the frequency is the number of revolutions per unit time. Then, equation (14) represent the simple harmonic motion with periodic time $\frac{2p}{w}$ and amplitude A.

3.7) circular motion

Consider a particle "p" moves on the circumference of circle with radius "a" and center "O" i.e. r=a then

$$\dot{r} = zero \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

let F_r and F_q are the components of the acting force in the directions of increasing r and q respectively. Then the equation of motion are

$$m\left(-a\dot{q}^2\right) = F_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

$$m\left(a\ddot{q}\right) = F_q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

3.7.1) Circular motion with constant angular velocity

when $q^{0} = w = constant$ ....(21)

eqs (19) and (20) takes the form

$$F_r = -maw^2 \quad \text{………..…………….………(22)}$$

$$F_q = 0 \quad \text{………..………...………………(23)}$$

i.e. the acceleration is in the direction of the radius vector towards the center of the circle.

Integrate eqs. (21) we get $q = wt + q_0$ ………………………………………(24)

Where q_0 is the angle at the beginning of the motion.

3.7.2) The periodic time

if q is the angle at time "t", then $(q + 2p)$ is the angle at time $(t + t)$

i.e. $(q + 2p) = w(t + t) + q$ …………………………..…(25)

subtract (24) from (25) we get $t = \dfrac{2p}{w}$ …………………..…..…………….(26)

3.7.3) The relation between the circular motion and S.H.M

Consider the Cartesian ***ox, oy*** in the plane of motion. Let "p" be any point has a coordinates

$$x = a\cos q = a\cos(wt + q_0)$$

$$y = a\sin q = a\sin(wt + q_0) = a\cos\left(wt + q - \tfrac{p}{2}\right) \text{……………………………….……………(27)}$$

i.e. the projection of point "p" on the axes ***ox, oy*** moves in a simple harmonic motion with amplitude "a" and periodic time t .

SOLVED EXAMPLES

1-A particle is moving in **S.H.M**, with periodic time **4** sec. if it starts from rest at a distance **4** ft from the center, find the time elapses before it has described **2** ft and it's velocity.

Solution

If the acceleration be w^2 times the distance, we have $\frac{2p}{w} = 4 \qquad \therefore w = \frac{p}{2}$

When the point has described 2 ft, it is then at a distance of 2 ft from the center of its motion. Hence the time that has elapsed = $\frac{1}{w} cos^{-1}\left(\frac{x}{A}\right) = \frac{2}{p} cos^{-1}\left(\frac{2}{4}\right) = \frac{2}{p}.\frac{p}{3} = \frac{2}{3}$ sec.

Also, the velocity $= w\sqrt{A^2 - x^2} = \frac{p}{2}\sqrt{4^2 - 2^2} = p\sqrt{3}$ ft/sec.

2-A point starts from rest at a distance of **16** ft from the center of its path and moves in **S.H.M**. in its initial position the acceleration be **4** ft/sec^2 find

(a) Its velocity at a distance of **8** ft from the center

(b) Its periodic time.

Solution

(i) let the acceleration be w^2 times the distance then $16(w^2) = 4$ i.e. $w^2 = \frac{1}{4}$

hence its velocity at a distance of 8 ft from the center = $\sqrt{\frac{1}{4}(16^2 - 8^2)} = \sqrt{48} = 4\sqrt{3}$ ft/sec.

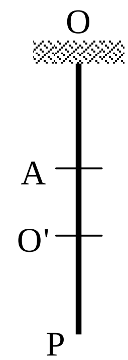
also its velocity when passing through the center = $\sqrt{\frac{1}{4}(16)^2} = 8$ ft/sec.

(ii) its periodic time = $\frac{2p}{w} = 4p$ sec.

3-A light spring whose unscratched length is (**ℓ**) cm, and whose modulus of elasticity is the weight of (**n**) grammas, is suspended by one end and has a mass of (**m**) grammas attached to the other, show that the time of a vertical oscillation of the mass is

$$t = 2p\sqrt{\frac{m\,\mathbf{l}}{n\,g}}$$

<u>Solution</u>



Let "O" be a fixed end of the spring, OA its position when unstreched ,

When the particle is at "P" ,where **OP =** ***x***

Let T be the tension of the spring then, hook's law

$$T = l\,\frac{x-\mathbf{l}}{\mathbf{l}} = ng\frac{x-\mathbf{l}}{\mathbf{l}}$$

hence the resultant upward force on "P" = *T- mg*

$$F = ng\frac{x-\mathbf{l}}{\mathbf{l}} - mg = \frac{ng}{\mathbf{l}}\left[x - \frac{m+n}{n}\mathbf{l}\right]$$

let " O' " be a point on the vertical through "O" such that $OO' = \frac{m+n}{n}\mathbf{l}$, Hence the resultant upward force on "P" = $= \frac{ng}{\mathbf{l}}[OP - OO'] = \frac{ng}{\mathbf{l}}[O'P]$

its motion is simple harmonic motion about O', as center and its time of oscillation is

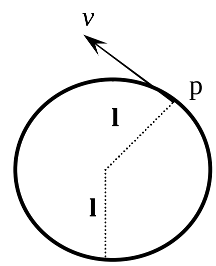
$$= 2p \,/ \sqrt{\frac{ng}{m\mathbf{l}}} = 2p\sqrt{\frac{m\mathbf{l}}{ng}}$$

it will be noted that " O' " is the point where the mass would hang at rest, if it were placed at rest at "P" the upward tension would be

$$= g\frac{[OO' - \mathbf{l}]}{\mathbf{l}} = \frac{ng}{\mathbf{l}}\left[\left(\frac{m+n}{n}\right)\mathbf{l} - \mathbf{l}\right] = mg$$

4-A particle is attached to one end of the ends of inelastic string of length l. The other end of the string is fixed at a point of a horizontal plane. When the string is tension, the particle is projected with horizontal initial velocity v_0 perpendicular to the direction of the string. Study the motion and find the tension in the string.

<u>Solution</u>



the acting force on the particle "p" is parallel to the radius.

i.e. the angular velocity is constant $q^{o} = w = constant$

then $q^{o} = \frac{v_0}{a}$

$T = -ma\frac{v_0^2}{a^2} = -\frac{mv_0^2}{a}$ The minus sign shows the direction of the tension.

The periodic time is $t = \frac{2p}{w} = \frac{2pa}{v_0}$

PROBLEMS

1- A particle moves in **S.H.M**, its velocities **6** cm/s and **8** cm/s when the particle is at distance **4** cm and **3**cm respectively from the center of motion. Find the amplitude of motion and frequency, also find the max. velocity and the max. acceleration.

2- A particle moves in **S.H.M** on straight line **AOB** with frequency (**1/π**) vib/sec. if **O** is the center of vibration. **A,B** are the external positions of the motion and P1, P2 are bisecting points of **OB** and **OA** respectively. Calculate the time needed for the particle arrive at those points.

3- a particle moves in straight line defined by $x = 3\sin 2t + 4\cos 2t$ prove that the motion is **S.H.M** then find the amplitude, the periodic time, maximum velocity.

4- A point **P** moves in a simple harmonic motion. If the distance of the point **P** from the center of the motion at the ends of three consecutive seconds are **1** cm, **5** cm, **5** cm measured in the same direction with respect to the center, find the periodic time.

5- A particle with mass **m** is suspended from a fixed point **A** by an elastic string of natural length ℓ_0 , and modulus of elasticity **4mg** , where **g** is gravity acceleration. if the particle is left to fall vertically from **A**, prove that the max. depth is $2\ell_0$ downward the point **A**. also prove that the time needed to cover this distance is

$$\sqrt{\frac{l_0}{g}}\left[\sqrt{2}+\frac{p}{4}+\frac{1}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

6- A particle of mass **1 kg** is suspended by a light elastic string such that its length increases to $2\ell_0$ cm. if the particle is attracted after then a distance ℓ_0 cm downward and leaves to move. Find the time needed for the particle to return to the state of rest again.

CHAPTER 4

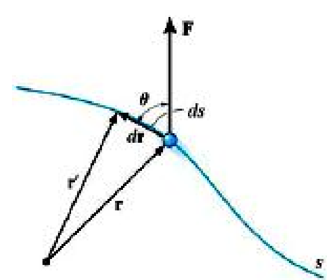
WORK & ENERGY

4.1) introduction

In this chapter we shall integrate the equation of motion with respect to displacement and thereby obtain the principle of work and energy. This principle is useful for solving problems which involve force, velocity, and displacement.

4.2.1) The work of a force

In mechanics a force **F** does work only when it undergoes a displacement in the direction of the force, for example, consider the force **F** having a location on the path S which is specified by the position vector $\vec{r}$, if the force moves along the path to a new position $\vec{r} = \vec{r} + d\vec{r}$; the displacement is then $d\vec{r}$.



The work dW which is done by $\vec{F}$ is a scalar quantity defined by

$$dW = \vec{F}.d\vec{r} = F\,dS\cos q\,,\text{where}\quad \left|\vec{dr}\right| = dS$$

i.e. the work is represented as the product of **F** and the component of displacement in the direction of the force i.e. $F\,dS\cos q$, or as the product of dS and the component of the force in the direction of displacement, i.e. $F\cos q$.

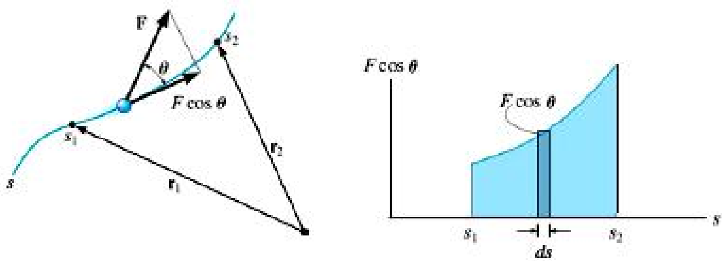
dW is +ve if $0^{o} \le q \le 90^{o}$ & dW is -ve if $90^{o} \le q \le 180^{o}$ & $dW = 0$ if $q = 90^{o}$ i.e. $\vec{F} \perp d\vec{r}$

or if the force is applied at a fixed point, in which the displacement is zero.

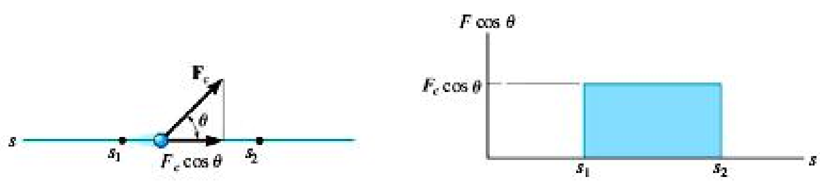
⊗ the basic unit for work in the SI system called a joule (j) = N.m.

4.2.2) The work of a variable force

if a force undergoes a finite displacement along its path from S_1 to S_2. and F = F(s), we have $W_{1-2} = \int_{s_1}^{s_2} F\cos q\, dS = \int_{r_1}^{r_2} \vec{F}\, d\vec{r}$

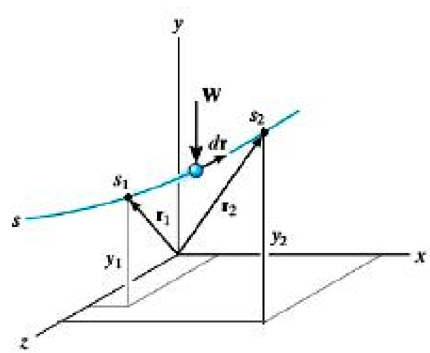


if the working component of the force $F\cos q$ is plotted versus S, as in figure the integral represented in this equation can be interpreted as the area under the curve between the points S_1 and S_2.



if the force Fc has a constant magnitude and acts at a constant angle q from its straight line path then, $W_{1-2} = F\cos q\,(S_2 - S_1)$ which is equal to the area of the rectangle.

4.2.3) work of a weight



consider a particle which moves up along the path **S** from point (1) to point (2). At an intermediate point, the displacement

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ since $Wt = -Wt\,\vec{j}$

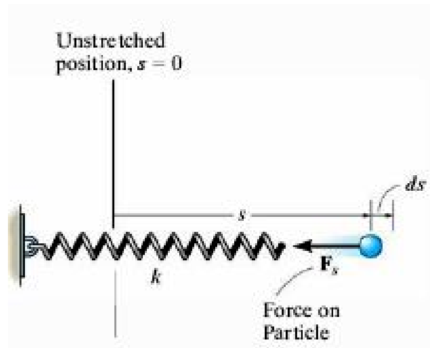
$$W_{1-2} = \int_{r_1}^{r_2} \vec{F}\, d\vec{r} = \int_{r_1}^{r_2} \left(-Wt\,\vec{j}\right) \cdot \left(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right) = \int_{y_1}^{y_2} -\vec{Wt}\, d\vec{y} = -Wt\left(y_2 - y_1\right) = -Wt\left(Dy\right)$$

however that if the particle is displaced downward $(-Dy)$ the work of the weight is **+ve**

4.2.4) work of a spring force

the magnitude of force developed in a linear elastic spring when the spring is displaced a distance S from its unstreched position is **Fs = K . S,** where **K** is the spring stiffness**.** If the spring is elongated or compressed from a position **S_1** to **S_2** then, the **work done on** the spring **by Fs** is (**+ve**) since in each case the force and displacement are in the same direction. We require

$$W_{1-2} = \int_{S_1}^{S_2} F\, dS = \int_{S_1}^{S_2} K.S\, dS = \tfrac{1}{2} K\left(S_2^2 - S_1^2\right)$$



If a particle (or body) is attached to the spring, then The force **Fs** exerted on the particle is opposite to that exerted on the spring, consequently the force will do **(-ve)** work on the particle

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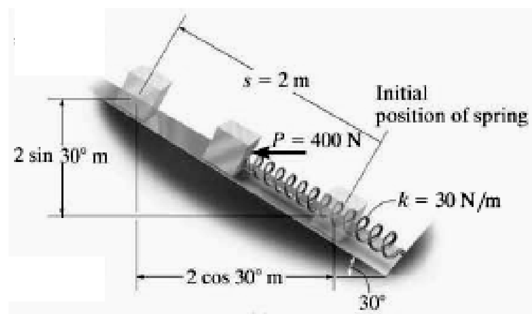
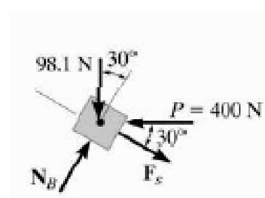
Example (1)

The 10-kg block shown in Fig. rests on the smooth incline. If the spring is originally unstretched, determine the total work done by all the forces acting on the block when a horizontal force **P = 400 N** pushes the block up the plane **s = 2** m.
(K=30N/m, θ=30º)



Solution



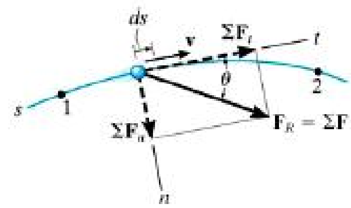
Work done by the force **p** = $p\, cos q\, (S)$ = 400(2cos30º)= 692.8 j

Work done by the spring force **Fs** = **-**$\frac{1}{2}K\left(S^2\right)$ = -½(30)(2)²= -60 j

Work done by the weight **mg** = $-mg(y)$ = **-** (10*9.81)(2sin30º)= -98.1 j

The total work done W= 692.8-60-98.1= 535 j

4.3) principle of work and energy



consider a particle "p" which is located at some arbitrary point on its path as shown. At the instant considered "p" is subjected to a system of external forces, represented by the resultant $F_R = \sum \vec{F}$. Normal components do no work.

Tangential components $\sum \vec{F_t} = m a_t$, since $d\vec{r}$ has a magnitude dS along the path.

$$\sum F \cos q = m a_t = m\left(\frac{dv}{ds}.\frac{ds}{dt}\right) = mv\left(\frac{dv}{ds}\right)$$

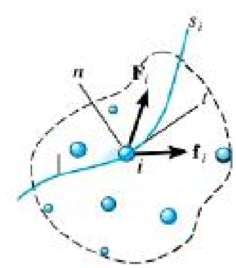
$$\therefore \int_{s_1}^{s_2} F \cos q \,(ds) = \int_{v_1}^{v_2} mv(dv)$$

$$\therefore W_{1-2} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = K_2 - K_1$$

Which is the principle of work and energy for the particle. It is convenient to rewrite it in the form $\therefore K_2 = K_1 + W_{1-2}$

Which states that the particle's initial kinetic energy plus the work done by all forces acting on the particle as it moves from its initial to its final position is equal to the particle's final kinetic energy.

4.4) principle of work and energy for a system of particles



Inertial coordinate system

consider a system of **n** particles isolated within an enclosed region of space as shown. Here the arbitrary **i** th particle, having a mass m_i is subjected to a resultant external forces F_i and a set of internal forces. $\sum_{i=1}^{n} F_{ij} = f_i$ which the other particles exert on the **i** th particle.

The principle of work and energy written for **i** th particle is thus

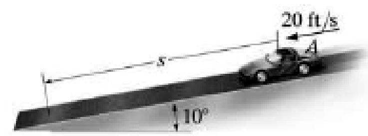
$$\therefore \int_{r_{i1}}^{r_{i2}} \vec{F_i}\, d\vec{r_i} + \int_{r_{i1}}^{r_{i2}} \vec{f_i}\, d\vec{r_i} = \tfrac{1}{2}m_i v_{i2}^2 - \tfrac{1}{2}m_i v_{i1}^2$$

$$\therefore \sum W_{1-2} = \sum K_2 - \sum K_1$$

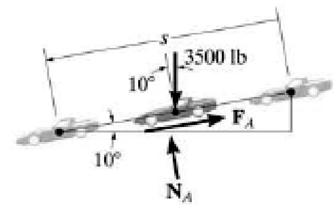
The internal work for a rigid body is zero.

Example (2)

The **3500-**lb automobile shown in Fig. travels down the **10°**inclined road at a speed of **20** ft/s. If the driver jams on the brakes, causing his wheels to lock, determine how far **s** the tires skid on the road. The coefficient of kinetic friction between the wheels and the road is $\mu = 0.5$.



Solution



$$\sum Fy = 0 \Rightarrow N_A - 3500(\cos 10^{\circ}) = 0$$

$$N_A = 3446.8(lb)$$

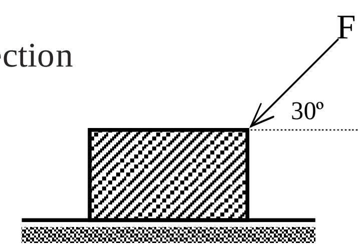
thus $F_A = \mu N_A = 0.5(3446.8) = 1723.4(lb)$

the principle of work and energy $\therefore K_2 = K_1 + W_{1-2}$

$$\frac{1}{2}\left(\frac{3500}{32.2}\right)(20)^2 + 3500(S \sin 10^{\circ}) - 1723.4S = 0 \qquad \Rightarrow S = 19.5\,ft$$

Example (3)

The **2-kg** block is subjected to a force having a constant direction and magnitude $F = \frac{300}{1+S}$ N, where **S** is measured in meter. When **S=4m**, the block is moving to the left with a speed of **8**m/s. determine its speed when **S=12** m/s. the coefficient of kinetic friction between the block and the ground is $\mu = 0.25$.



Solution

$$\because W_{1-2} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \qquad \Rightarrow \int_{s_1}^{s_2} (F\cos 30^{\circ} - \mu N) = \frac{1}{2}m(v_2^2 - v_1^2)$$

$$\Rightarrow \int_{s_1}^{s_2} \left(\left(\frac{300}{1+S}\right)\cos 30^{\circ} - 0.25\left(\frac{300}{1+S}\sin 30^{\circ} + 19.6\right)\right) ds = \frac{1}{2}(2)(v_2^2 - (8)^2)$$

$$\therefore v_2 = 15.4\,m/\sec$$

4.5) Conservation forces and potential energy

if the total work done by forces in performing a series of displacement which bring the bodies to their original position is zero, such forces are called conservation forces; for example the work done by gravity when a particle of mass **m** ascends height h and then comes back to the surface of the earth is –mg (h) +mg (h)=0.

Such is not the case with all forces. If for example a body is dragged through a distance **S** against a constant frictional force **F**, the work done is **F.S** to bring the body to its former position on the same path an equal moment of work **F.S** is to be done again. Thus the total work performed to bring the body its original position is 2 F.S (or more depending on the path followed by the body) and not zero. These second type of forces are called non-conservative.

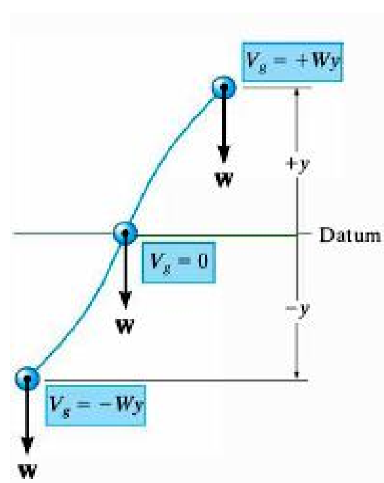
4.5.1) potential energy

the potential difference between two points **a, b** is defined as the work done exerted against the effective force, in moving a body from point **a** to point **b**, i.e.

$$\therefore W_{a-b} = -(U_b - U_a) = -\int_a^b \vec{F} \cdot d\vec{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

the potential energy $U(r)$ is a scalar function depends only on the position of a point $\vec{r} = (x,y,z)$ and explicit independent on time.

a) Gravitational potential energy

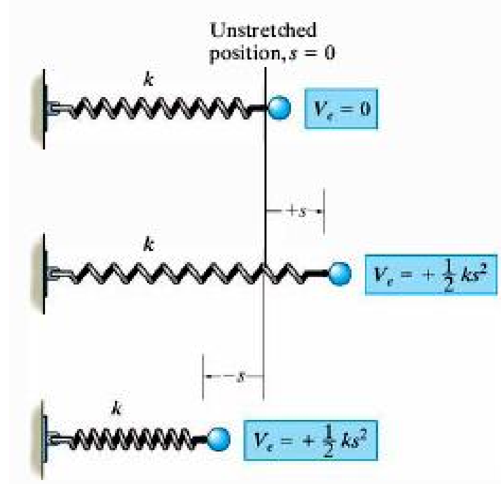


if a particle is located a distance **y** above an arbitrarily selected datum as in figure, the particle's weight W_t has positive U_g since W_t has the capacity of doing positive work when the particle is moved back down to the datum. Likewise, if the particle is located a distance **y** below the datum, U_g is negative since the weight does negative work when the particle is moved back up to the datum. At the datum $U_g = 0$;

in general if **y** is **+ve** upward, the G.P.E of the particle of weight W_t is thus $U_g = W_t\,(y)$.

b) Elastic potential energy

when an elastic spring is elongated or compressed a distance **S** from its unstretched position, the E.P.E. U_e due to spring is $U_e = \frac{1}{2} KS^2$, i.e. **always positive**.



⊗ if a particle is subjected to both gravitational and elastic forces, the particle's potential energy is $U = \pm mg(y) + \frac{1}{2} KS^2$.

4.5.2) Relation between work done and energy

from the relation between work done energy $(K_b - K_a) = W_{a-b} = -(U_b - U_a) =$ ………………(2)

if the L.H.S of (2) is +ve then the R.H.S is also positive this means that the increases in kinetic energy of a body between two positions is accompanied by decrease in potential energy. Or

$$(K_a + U_a) = (K_b + U_a) = constant \quad \text{………………………………(3)}$$

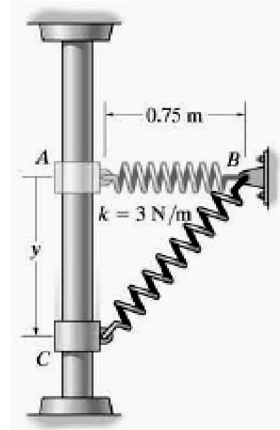
which means that the sum of kinetic and potential energy at any point is always constant independent on time.

The last formula represent the law of conservation of energy which states that the total energy for a system of bodies is constant, if the forces exerted are conservative i.e. does not depend on the time explicitly.

⊗ the gravity, electromagnetic the nuclear,... forces are conservative, but the frictional forces are non-conservative.

Example (4)

A smooth **2**-kg collar, shown in Fig., fits loosely on the vertical shaft. If the spring is unstretched when the collar is in the position **A**, determine the speed at which the collar is moving when **y = 1** m, if (**a**) it is released from **rest** at **A**, and
(**b**) it is released at **A** with an velocity **V_A = 2** m/s.



Solution

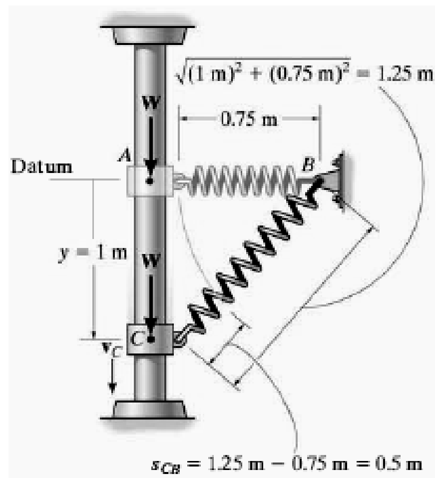
(a) the collar is released from rest at A, and

$$(K_A + U_A) = (K_C + U_C)$$

$$0 + 0 = \tfrac{1}{2} m v_c^2 + \left\{\tfrac{1}{2} k s^2 - mg(y)\right\}$$

$$0 + 0 = \tfrac{1}{2} 2 v_c^2 + \left\{\tfrac{1}{2} 3(0.5)^2 - 2 * 9.8(1)\right\}$$

$$\underline{v_c = 4.39\, m/s}$$



(b) the collar is released with $\underline{v_A = 2\, m/s}$

$$(K_A + U_A) = (K_C + U_C)$$

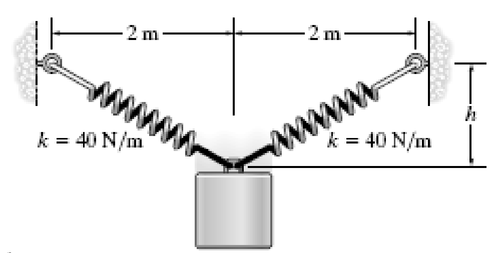
$$\tfrac{1}{2} m v_A^2 + 0 = \tfrac{1}{2} m v_c^2 + \left\{\tfrac{1}{2} k s^2 - mg(y)\right\}$$

$$\tfrac{1}{2} 2(2)^2 + 0 = \tfrac{1}{2} 2 v_c^2 + \left\{\tfrac{1}{2} 3(0.5)^2 - 2 * 9.8(1)\right\}$$

$$\underline{v_c = 4.82\, m/s}$$

Example (5)

The cylinder has a mass of **20** kg and is released from **rest** when (**h=0**). Determine its speed when (**h=3m**).The springs each have an unstretched length of **2** m.



Solution

$$(K_A + U_A) = (K_B + U_B)$$

$$0 + 0 = \tfrac{1}{2} m v_B^2 + \left\{\tfrac{1}{2} k s^2 - mg(y)\right\}$$

$$0 + 0 = \tfrac{1}{2}(20) v_B^2 + 2\left\{\tfrac{1}{2} 40\left(\sqrt{2^2 - 3^2} - 2\right)^2\right\} - 2 * 9.81(3)$$

$$\underline{v_B = 6.97\, m/s}$$

4.5.3) conservative force

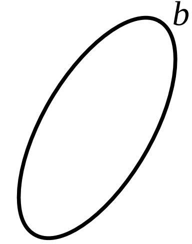
Since $U(r)$ is a function of position only and independent on the time then $\vec{F}$ is a function of position only.

And since the work done exerted by $\vec{F}$ in displacement a body from a to b independent on the path, then

$$\therefore W(a \to b) = -W(b \to a)$$

$$\therefore W(a \to b) + W(b \to a) = 0$$

i.e. $$\oint \vec{F} . \vec{dr} = 0$$



4.5.4) Relation between the potential energy and conservative force

since $U(b) - U(a) = -W(a \to b) = -\int_a^b \vec{F} . \vec{dr}$ then the potential energy for a body at a point $\vec{r}$ is $U(r) = U(a) - W(a \to b) = U(a) - \int_a^r \vec{F} . \vec{dr}$ where $U(a)$ is constant and is not defined. If we choose $U(\infty) = 0$, then $U(r) = -\int_\infty^r \vec{F} . \vec{dr}$ which means that the potential energy of a body at position $\vec{r}$ equal the work done exerted against the effected force to remove it from ∞ to $\vec{r}$.

From the last formula and using the Cartesian product of vectors we have $dU(x, y, z) = -\vec{F} . \vec{dr}$

$$\left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial y}\right)dy = -(fx\, dx + fy\, dy + fz\, dz)$$

or $$fx = -\left(\frac{\partial u}{\partial x}\right) \; , fy = -\left(\frac{\partial u}{\partial y}\right) \; , fz = -\left(\frac{\partial u}{\partial z}\right)$$

or $$\vec{F} = -\left[\left(\frac{\partial u}{\partial x}\right)\vec{i} + \left(\frac{\partial u}{\partial y}\right)\vec{j} + \left(\frac{\partial u}{\partial z}\right)\vec{k}\right] = -\vec{\nabla} U$$

or $$\vec{F} = -grad(U)$$

The last equation relates a force to its potential function $U(r)$ and thereby provides a mathematical criterion for proving that $\vec{F}$ is **conservative.**

Example (6)

prove that the force $F=\frac{l}{r^3}.\vec{r}$ is conservative, such that l is constant. Then find its potential function

Solution

If we take any closed path say a circle of radius "C" and origin as a center, then the equation of the circle $\underline{r^2=C^2}$ or $\underline{\vec{r}.\vec{r}=C^2}$ $\Rightarrow$ $\underline{\vec{r}.\vec{dr}=0}$ then $\oint \vec{F}.\vec{dr}=0$

$\oint \frac{l}{r^3}.\vec{r}.\vec{dr}=0$ therefore this force is conservative.

To find the potential function $U(r)$, $U(r)=-\int_{\infty}^{r}\vec{F}.\vec{dr}=-\int_{\infty}^{r}\frac{l}{r^3}.\vec{r}.\vec{dr}=-l\int_{\infty}^{r}\frac{dr}{r^2}=\frac{l}{r}$

Example (7)

Prove that the force $\vec{F}=\left(3yz^2\right)\vec{i}+\left(3xz^2+z\right)\vec{j}+(6xyz+y)\vec{k}$ is conservative, then find its potential function

Solution

Since $\frac{\partial Fx}{\partial y}=3z^2$, $\frac{\partial Fy}{\partial x}=3z^2$ and $\frac{\partial Fx}{\partial z}=6yz$, $\frac{\partial Fz}{\partial x}=6yz$ and $\frac{\partial Fy}{\partial z}=6xz+1$, $\frac{\partial Fz}{\partial y}=6xz+1$

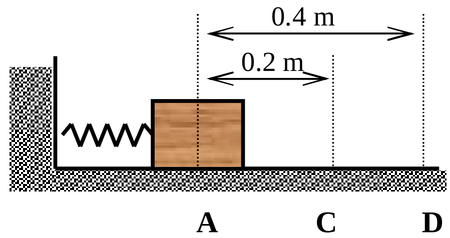
Therefore this force is **conservative**.

To find the potential function $U(r)$, $U(r)=-\int_{\infty}^{r}\vec{F}.\vec{dr}=-\int_{\infty}^{r}\left[\left(3yz^2\right)dx+\left(3xz^2+z\right)dy+(6xyz+y)dz\right]$

$$U(r)=-\left[\left(3xyz^2\right)+\left(3xyz^2+yz\right)+\left(3xyz^2+yz\right)\right]$$

$$\therefore U(r)=-\left[\left(3xyz^2+yz\right)+C\right]$$

PROBLEMS



1- 10-kg block rests on the horizontal surface. A spring which in not attached to the block has a stiffness **K= 500** N/m and is initially compressed **0.2m** from **C** to **A**, after the block is released from **rest** from **A**, determine its velocity when it passes point **D**. The coefficient of friction between the block and the plane is $m = 0.2$

2- A variable force $\vec{F} = 2y\,\vec{i} + xy\,\vec{j}$ acts on a particle. Find the work done by the force when the particle is displaced from the origin to the point **p**=(4,2) along the parabola $y^2 = x$.

3- A body of mass **4-kg** moves in the force field $F = \frac{200}{r^3}.\vec{r}$

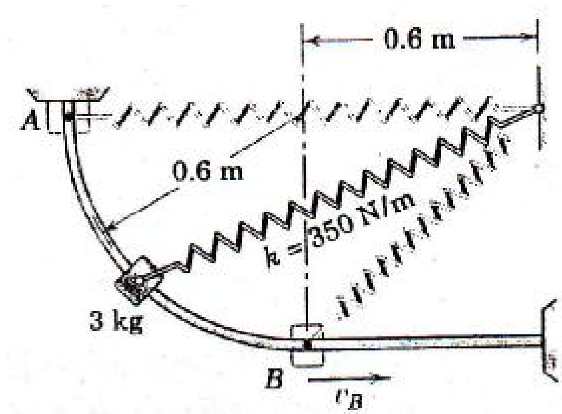
(i) prove that this force is conservative

(ii) find the potential function.

(iii) if the velocity of the body $V(r=1) = 20\,m/s$, find its velocity art $r = 2m$.

4- Find the conservative force which has the potential function $U(x,y,z) = 3x^2z^2 - 2xy^2z^3 + C$ then find the work done by this force to move the body from point A=(-2,1,3) to point B=(1,-2,-1).

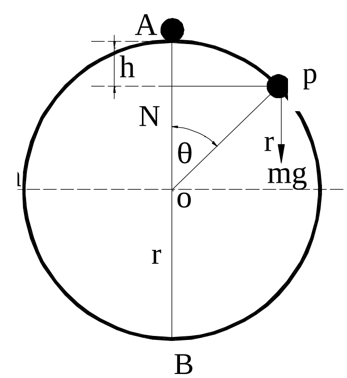
5- A slider of mass **3** kg attached to a spring of stiffness **350** N/m and unscratched length **0.6** m is released from **rest** at A as shown . Determine the velocity of the slider as it passes through **B** .



CHAPTER 5

Motion along A Smooth Vertical Circle

5.1) Motion on the outside of a smooth vertical circle.



Let a particle starts from rest at the hightest point and moves down along the outside of the arc of a smooth circle, discuss the motion.

AOB is the vertical diameter of the circle of radius "**r**", p is any Position of the particle.

Let **AN = h = AO-ON = r (1-cosθ)** …………………..(1)

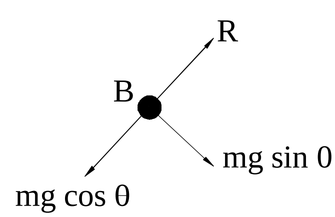
From the law of conservation of energy

$$(K_A + U_A) = (K_p + U_p)$$

$$0 + mg(r) = \tfrac{1}{2}mv^2 + mg(ON) \qquad v^2 = 2g(r - ON) = 2gh \;\ldots\ldots\ldots\ldots.(2)$$

Equation of motion along the normal PO.

$$\frac{mv^2}{r} = mg\cos\theta - R$$



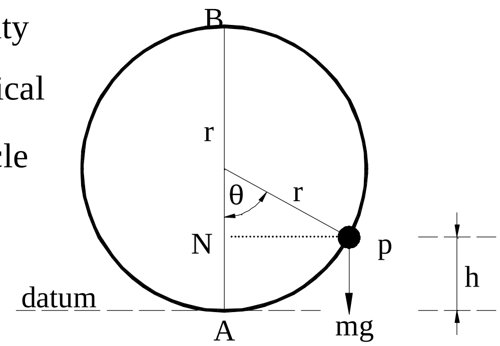
$$R = m\left(g\cos\theta - \frac{v^2}{r}\right) \quad \text{or} \quad R = mg(3\cos\theta - 2)$$

or $$R = \frac{mg}{r}(r - 3h) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

form (3) R vanishes when or $h = \frac{r}{3}$ or $\cos\theta = \frac{2}{3}$, the particle no longer passes the arc and leave it.

5.2) Motion on the inside of a smooth vertical circle.

A particle is projected from the lowest point with velocity v_0 and moves along the inside of the arc of smooth vertical circle, discuss the motion. **P** is any position of the particle at any time "t".



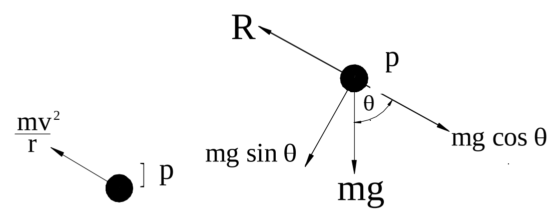
h=AN= r (1-cosθ)

From the law of conservation of energy

$$(K_A + U_A) = (K_p + U_p)$$

$$\tfrac{1}{2} m v_0^2 + 0 = \tfrac{1}{2} m v_p^2 + mg(AN) \qquad \underline{v^2 = v_0^2 - 2gr(1 - \cos\theta)} \ldots\ldots\ldots\ldots(1)$$

Equation of motion along the normal PO.



$$\frac{mv^2}{r} = R - mg\cos\theta$$

$$R = m\left(\frac{v^2}{r} + g\cos\theta\right) \quad \text{or} \quad \underline{R = \frac{mv_0^2}{r} + mg(3\cos\theta - 2)} \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Equation (1) gives the velocity of the particle and equation (2) gives R, pressure of the curve on the particle.

⊗ at the lowest point A, $h = 0, (\theta = 0)$ and $R = \frac{mv_0^2}{r} + mg$ ………….(3)

⊗ at the highest point B, $h = 2, (\theta = \pi)$ and $R = \frac{m}{r}(v_0^2 - 5gr)$ and by (1) $v^2 = v_0^2 - 4gr$

in order that the particle may not leave contact with the arc of the circle, **R** should not vanish till the particle reaches the highest point, i.e. $v_0^2 > 5gr$,the velocity v also does not vanish at any point; thus the particle makes a complete revolution.

when $v_0^2 = 5gr$, then from (3) $R = 6mg$ so that if the particle is projected with a velocity just sufficient to take it to the highest point, pressure at the lowest point = 6mg , six times the weight of the particle.

Special cases:

1) Suppose that the velocity v vanishes at a height h_1 and the pressure R vanishes at a height h_2, then from (1) and (2) we have

$$h_1 = \frac{v_0^2}{2g} \text{ and } h_2 = \frac{v_0^2 + gr}{3g}$$

if $h_1 < h_2 \quad \Rightarrow \quad \frac{v_0^2}{2g} < \frac{v_0^2 + gr}{3g} \qquad \Rightarrow \quad v_0^2 < 2gr$

thus if the velocity of projection v_0 < the velocity sufficient to take the particle to the level of the horizontal diameter, the velocity vanishes before the pressure vanishes. The particle, therefore stop, and moves down acquires some velocity at the lowest point A, moves on the other side of A through an equal height and goes on moving to and from just like the bob of a clock pendulum.

2) if $h_1 < h_2 \quad \Rightarrow \qquad v_0^2 > 2gr$ so that if $v_0^2 > 2gr$, and $v_0^2 < 5gr$ i.e. when the velocity of projection is sufficient to take the particle higher than the level of the horizontal diameter and is insufficient to allow the particle to make complete revolutions, the pressure vanishes before the velocity vanishes. The particle therefore leaves the arc and on account of the velocity it possesses, it moves freely along a parabolic path whose equation can be obtained.

3) Bead on a smooth vertical circular wire, or that of a particle moving inside a smooth vertical circular tube, the bead or the particle necessarily keeps to the circular path and the question of its leaving the circle does not arise.

4) Motion of a particle attached to the end of string. If a particle is hanging from a fixed point by a light inextensible string and is projected with a certain horizontal velocity, the motion is exactly the same as that of a particle moving inside a smooth vertical circle. We have only to substituting the tension (T) for the pressure (R), and the length of the string (l) for the radius (r) of the circle.

5) In the case when $v_0^2 > 2gr$, and $v_0^2 < 5gr$, the tension vanishes some where above the point of suspension, the string becomes **slack** and the particle describes a parabola freely so long as the string does not become tight again.

6) In order that the particle may make complete revolutions, the string must be strong enough to bear a tension of at least **six** times the weight of the particle.

Example (1)

A heavy ring is constrained to move on a smooth vertical circle wire of radius (a), if it is projected from the lowest point by velocity $\sqrt{3ga}$, Find

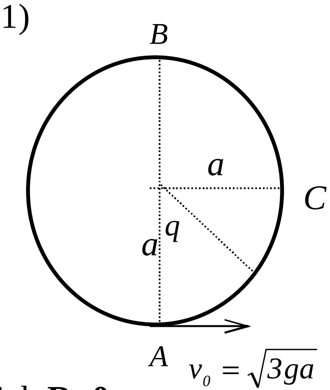
(i) The angle at which the ring change its contact with wire and its velocity at this point.

(ii) The max. height at which the ring reaches with this velocity.

(iii) The least velocity which the ring can be projected to describe a complete revolution.

solution

(**i**) since $v^2 = v_0^2 - 2gr(1 - \cos\theta) \;\Rightarrow\; v^2 = ga(1 + 2\cos\theta)$ ……………(1)



The equation of motion along the radius.

$$\frac{mv^2}{r} = R - mg\cos\theta \text{ …………..(2)}$$

from (1) and (2) $R = mg(1 + 3\cos\theta)$………..(3)

the bead changes its position of contact with wire at a point for which **R=0**

$$0 = mg(1 + 3\cos\theta) \Rightarrow \theta = \cos^{-1}\left(\frac{-1}{3}\right)$$

the velocity at this instant $v^2 = ga\left(1 + 2\left(\frac{-1}{3}\right)\right) \Rightarrow v = \sqrt{\frac{ga}{3}}$

(**ii**) to find the max. height, we know that the velocity at the max height is zero

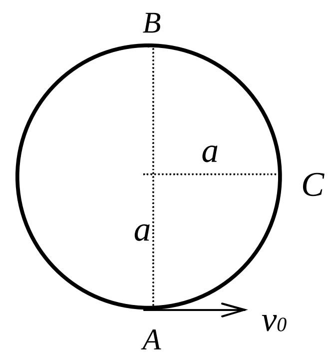
since $v^2 = ga(1 + 2\cos\theta)$ or $v^2 = v_0^2 - 2gh \quad \Rightarrow \quad 0 = (3ga) - 2gh \qquad \therefore h = \frac{3}{2}a$

(**iii**) from (1) put $\theta = 180^\circ$ and $v^2 \geq 0$

$$v_0^2 \geq 2ga(1 - \cos 180^0) \;\Rightarrow\; v_0^2 \geq 4ga \Rightarrow \; v_0 \geq 2\sqrt{ga}$$

Example (2)

A Stone of Weight **W** is tied to one end of a string and is describing a circle in a vertical plane, if the maximum speed of the stone is **twice** the minimum speed, Prove that when the string is horizontal its tension is **10W/3**.



solution

since $v_{max} = 2v_{min} \Rightarrow \underline{\therefore v_A = 2v_B}$ ....(1)

$$v_B^2 = v_A^2 - 2ga(1 - \cos p) = v_A^2 - 4ga$$

$$\frac{3}{4}v_A^2 = 4ga \Rightarrow v_A^2 = \frac{16ga}{3} \quad \text{.........................(2)}$$

The equation of motion along the radius OC.

At OC, $q = \frac{p}{2} \Rightarrow \frac{mv_C^2}{a} = R - mg\cos\frac{p}{2}$

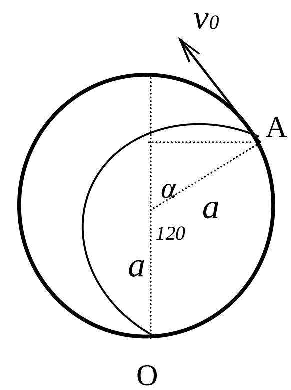
$$\therefore R = \frac{mv_C^2}{a} \quad \text{....................(2)}$$

to get the velocity at point C $v_C^2 = v_A^2 - 2ga\left(1 - \cos\frac{p}{2}\right) \Rightarrow v_C^2 = \frac{16}{3}ga - 2ga = \frac{10}{3}ga$ ......(3)

substituting (3) into (2) we get $\therefore R = \frac{m}{a}\left(\frac{10}{3}ga\right) = \left(\frac{10}{3}\right)mg = \left(\frac{10}{3}\right)W$

<u>Example (3)</u>

A Particle hanging from a fixed point by a light string of length (a) is projected horizontally with velocity $v_0 = \sqrt{nga}$ to describe a vertical circle, find the value of ***n*** that makes the string **slack** at an angle **θ = 120**°, then show that the parabolic path of the particle passes through its initial point of projection.



<u>solution</u>

<u>The equation of motion along the radius.</u>

$$\frac{mv^2}{a} = R - mg\cos\theta \quad \ldots\ldots\ldots\ldots(1)$$

since the particle leaves its contact (**slack**) at **θ = 120**. therefore **R=0**

$$\therefore \frac{mv^2}{a} + mg\cos\theta = 0 \quad \Rightarrow \quad \therefore \frac{m(nga)}{a} + mg\cos\left(\frac{2\pi}{3}\right) = 0 \quad \Rightarrow \quad \underline{\therefore n = \frac{7}{2}} \qquad \text{and} \quad v_0^2 = \frac{7}{2}ga$$

The path of the particle after falling is a parabola (motion of projectile)

$$y = x\tan\alpha - \frac{g\,x^2}{2v_0^2\cos^2\alpha} \ldots\ldots\ldots\ldots\ldots(2)$$

to substitute in equation (2), we first find the velocity of projection at point "A"

$$v^2 = v_0^2 - 2gr(1-\cos\theta) \quad \Rightarrow \quad v_A^2 = \frac{7}{2}ga - 2ga(1-\cos 120^{\circ}) = \frac{1}{2}ga$$

by substituting $\alpha = 60^0$, $x = \frac{\sqrt{3}}{2}a$, $y = -\frac{a}{2}$

$$\therefore y_0 = \left(\frac{\sqrt{3}}{2}a\right)\tan 60^0 - \frac{g\left(\frac{3}{4}a^2\right)}{2\left(\frac{1}{2}ga\right)(\cos 60^{\circ})^2} = \frac{3}{2}a - 3a = -\frac{1}{2}a$$

since $y_0 = y$, therefore the parabolic path of the particle passes through its initial point of projection.

PROBLEMS

1- A ball of mass **10**-kg is attached to one end of a string and is describing a circle in a vertical plane, if the maximum speed of the ball is twice the minimum speed, Find the tension when the string makes an angle of **45**° with the downward vertical.

2- A particle of mass **5** lb constrained to move inside a circle of radius **3** ft, if it is projected horizontally with speed **25** ft/sec, find

a) the velocity and the reaction when the string be horizontal.

b) the least velocity of projection that the particle may be able to make a complete revolution.

c) the max. tension that the string must be able to bear .

3- A particle is projected horizontally with speed $\sqrt{\frac{1}{2}ga}$ from the highest point of the outside of a fixed smooth sphere of radius (a), show that it will leaves the sphere at point whose vertical distance below the point of projection is $\left(\frac{a}{6}\right)$.

4- A particle attached to a string of length (**ℓ**), it is projected horizontally with a velocity $v_0^2 = (2+\sqrt{3})g\mathbf{l}$, show that it will rise to height of $\mathbf{l}\left(1+\frac{1}{\sqrt{3}}\right)$ before the string becomes **slack**, then find its velocity at this position.

5- A particle attached to a string of length (a), it is projected horizontally with a velocity $v_0^2 = (2+\sqrt{3})ga$, show that the string becomes **slack** when it has described an angle $cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

CHAPTER 6

IMPULSE & MOMENTUM

6.1) introduction

In this chapter we will integrate the equation of motion with respect to time and thereby obtain the principle of impulse and momentum. This principle is useful for analyzing problems of impact, steady fluid flow, and system which gain or lose mass.

6.2) IMPULSE

Is defined as the product of the force and the length of the time interval. If $\vec{F}$ is a constant force acting on a certain body for a certain interval time "t" then the impulse is given by

$$\vec{P} = \vec{F}.t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In the case of variable force, the impulse is given by

$$\vec{P} = \int_{t_1}^{t_2} \vec{F}.dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The impulse vector acts in the same direction as the force, ans its magnitude has units of force-time (N.s) or (lb.s)

6.3) Linear Momentum

Each of the two vectors of the form $\vec{L} = m.\vec{v}$ is defined as the linear momentum of the particle. Since m is a scalar, the linear momentum vector $\vec{L}$ as the same direction as $\vec{v}$, and its magnitude mv has units of mass-velocity (kg.m/s) or (slug.ft/s).

6.4) Impulse – Momentum principle

the change in momentum of a particle during a time interval is equal to the net impulse exerted by the external forces during this interval.

$$\because \vec{F} = m\left(\vec{a}\right) = m\left(\frac{d\vec{v}}{dt}\right)$$

$$\therefore \int_{t_1}^{t_2} \vec{F}\,dt = m\int_{v_1}^{v_2} d\vec{v}$$

or Impulse $\vec{P} = m(v_2 - v_1)$ ....(3)

hence, change in momentum is equal to the impulse.

Scalar equations:

$$P\,x = \int_{t_1}^{t_2} \vec{F}x\,.dt = m\left[(v_x)_2 - (v_x)_1\right],\;\; P\,y = \int_{t_1}^{t_2} \vec{F}y\,.dt = m\left[(v_y)_2 - (v_y)_1\right],\; P\,z = \int_{t_1}^{t_2} \vec{F}z\,.dt = m\left[(v_z)_2 - (v_z)_1\right]$$

hence, change in momentum is equal to the impulse. For a system of particles moving under the effect of several external forces, we have:

$$\sum \vec{F}_i = \sum m_i\left(\frac{d\vec{v}_i}{dt}\right) \text{...............(4)}$$

i.e. the change of linear momentum of the system of particles is equal to the impulses of all the external forces acting on the system during the time t_1 to t_2.

6.4.1) Conservation of Linear momentum for a system of particles.

When the sum of the external impulses acting on a system of particles is zero, we have

$$\sum m(v_i)_2 = \sum m(v_i)_1 \text{ ....(5)}$$

this equation is referred to as the conservation of linear momentum. It states that the vector sum of the linear momentum for a system of particles remains constant through the time period t_1 to t_2.

6.4.2) Impulsive force

when a moving body strikes a fixed object or impinge on another there is a sudden change of motion. The forces acting on the bodies are great and act on them for such a short time that it is difficult to estimate either their intensity or the time during which they act. Such forces are known as impulsive forces. In these cases instantaneous changes of velocities take place and it is comparatively easy to measure the effect of the forces by their impulses or changes of momentum produced.

It is important to remember that an impulse is not a force

For example: a hammer strikes a ball of mass "**m**" and sends it off with a velocity "**u**" then **mu** is the magnitude of the impulse which is given to the ball and nothing can be knowing about the force exerted by the hammer unless the time during which the force acts is also known.

6.5) Impact (Collision of elastic bodies)

Elasticity : If we dropped a ball of glass on a marble floor, it rebounds almost to its original height but if the same ball were dropped on to a wooden floor, the distance through which it rebounds is much smaller. If further we allow an ivory ball and a wooden ball to drop from the same height up on a hard floor the height which they rebound are quite different. the velocities of these balls are the same when they reach floor, but since they rebound to different height their velocities on leaving the floor are different.

Again, when a ball strikes against a floor or when two balls of any hard material collide, the balls are slightly compressed and when they tend to recover their original shape, they rebound.

The property of the bodies which causes these differences in velocities and which makes them rebound after collision is called elasticity.

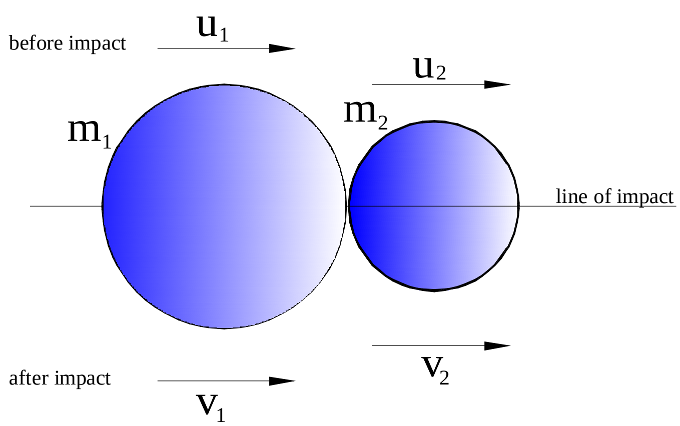
If a body does not tend to return to its original shape and does not rebound after collision, it is said to be inelastic.

In considering impact of elastic bodies, we suppose that they are smooth, so that the mutual action between them takes place only in the direction of their common normal at the point where they meet, there being no force in the direction perpendicular to their common normal.

When the direction of each body is along the common normal at the point where they touch, the impact is said to be direct, otherwise is said to be indirect or oblique.

6.5.1) Direct Impact

suppose two smooth spheres of masses m_1 and m_2 moving in the same straight line with velocities u_1 and u_2, impinges together. The forces which act between them during the collision act equally but in opposite directions on the two spheres so that the total momentum of the spheres remain unaltered by the impact. If U be the common velocity of the spheres after the collision and if the velocities are all measured in the same direction, we have



$$(m_1 + m_2)U = m_1 u_1 + m_2 u_2$$

this equation is sufficient to determine the one unknown quantity U .

But we know, as a matter of ordinary experience, that when two bodies of any hard material impinge on each other, they separate almost immediately and a finite change of velocity is generated in each by their mutual action depending on the material of the bodies. Hence the sphere, if free to move, will have after impact different velocities say v_1 and v_2

The equation of momentum now becomes

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

this single equation is not sufficient to determine the two unknown quantities v_1 and v_2. Another relation between the velocities is supplied by **Newton's Experimental** law which states that

"*when two bodies impinge directly, their relative velocity after impact is constant ratio to relative velocity before impact and is in the opposite directions*"

if bodies impinge obliquely, the same fact holds for their component velocities along the common normal at the point of contact.

The equation derived from this law for the above spheres is

$$-e = \frac{v_2 - v_1}{u_2 - u_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The constant ratio e is called the coefficient of elasticity or restitution. It depends on the substance of which the bodies are made and is independent of the masses of the bodies and their velocities before impact. The value of e differs considerably for different bodies and varies from zero to one.

- When (e=0), the bodies are said to be inelastic. In this case we have $v_1 = v_2$, i.e. if two inelastic spheres impinge they move with the same velocity after impact.
- When (e=1), the bodies are said to be perfectly elastic.

Both these are ideal cases never actually realized in nature.

To find the velocities of the spheres after direct impact we solve equations (1) and (2) we get

$$v = \frac{m_1u_1 + m_2u_2 - em_2(u_1 - u_2)}{m_1 + m_2} \quad \text{and} \quad v = \frac{m_1u_1 + m_2u_2 + em_2(u_1 - u_2)}{m_1 + m_2}$$

- ***Losses in kinetic energy***

In general there is always a loss of kinetic energy whenever two bodies impinge.

$$\text{Loss in K.E} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$= \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(u_1 - u_2)^2(1-e^2)$$

Example (1)

A ball of mass **8-**lb moving with velocity of **4** ft/s is overtaken by a ball of mass **12-**lb moving with a velocity of **9** ft/s. if (e =0.2), find the velocities of the balls after impact and the loss of kinetic energy

(**i**) if the two balls move in the same direction.

(**ii**) if the two balls move in the opposite direction.

Solution

(**i**) let the direction of motion of the first ball be taken as positive and let v_1, v_2 be the velocities after impact, then

$$8v_1 + 12v_2 = 8x4 + 12x9 = 140 \quad \text{and} \quad v_1 - v_2 = -0.2(4-9) = 1$$

which give $v_1 = 7.6$ ft/s $\quad v_2 = 6.6$ ft/s

$$\text{loss of K.E} = \left[\frac{1}{2}8(4)^2 + \frac{1}{2}12(9)^2\right] - \left[\frac{1}{2}8(7.6)^2 + \frac{1}{2}12(6.6)^2\right] = 57.6 \text{ ft.lb}$$

(**ii**)

$$8v_1 + 12v_2 = 8x4 - 12x9 = -76 \quad \text{and} \quad v_1 - v_2 = -0.2(4+9) = -2.6$$

which give $v_1 = -5.36$ ft/s $\quad v_2 = -2.76$ ft/s

$$\text{loss of K.E} = \left[\frac{1}{2}8(4)^2 + \frac{1}{2}12(9)^2\right] - \left[\frac{1}{2}8(5.36)^2 + \frac{1}{2}12(2.76)^2\right] = 389.3 \text{ ft.lb}$$

Example (2)

A ball **A**, moving with velocity of u_1 impinges directly on an **equal** ball **B** moving with a velocity of u_2 in the opposite direction. if A be brought to rest by the impact, show that $(1-e)u_1 = (1+e)u_2$

Solution

Let v_2 be the velocity of **B** after impact and let m be the mass of each, then since **A** is reduced to rest after impact, we have

$$m(0) + mv_2 = mu_1 - mu_2 \quad \text{or} \quad v_2 = u_1 - u_2$$

and $$0 - v_2 = -e(u_1 + u_2) \quad \text{or} \quad v_2 = e(u_1 + u_2)$$

from (1),(2) $$v_2 = u_1 - u_2 = e(u_1 + u_2)$$

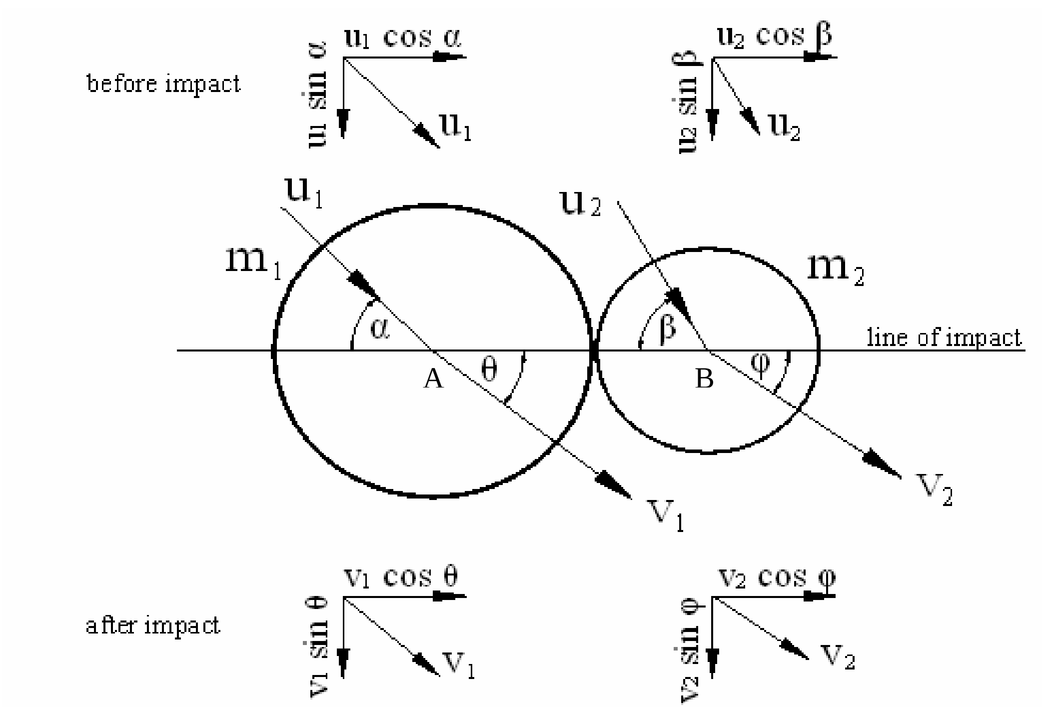
$$\therefore \quad u_1(1-e) = u_2(1+e)$$

6.5.2) In-Direct Impact [Oblique]

suppose that at the moment of impact the direction of motion of the spheres is not along the line joining their centers. let m_1 and m_2 be the masses of two smooth spheres with centers **A** and **B** at the time of impact, u_1 and u_2 the velocities just before impact and v_1, v_2 the velocities just after impact.

a , b the directions angles before impact and q , j the directions angles after impact.

Since the spheres are smooth, there is no impulse perpendicular to the line of centers and hence the resolved parts of velocities of the two spheres in the direction perpendicular to AB (line of impact) remain unaltered.



$v_1 \sin q = u_1 \sin a$ ………………………………………….(1)

$v_2 \sin j = u_2 \sin b$ …..……………………………………(2)

since the impulsive forces acting during the collision on the two spheres along their line of centers are equal and opposite, the total momentum along AB remains unchanged.

$$m_1 v_1 \cos\theta + m_2 v_2 \cos\varphi = m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

by Newton's experimental law for relative velocities resolved along the common normal AB, we have

$$v_1 \cos\theta - v_2 \cos\varphi = -e\left(u_1 \cos\alpha - u_2 \cos\beta\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

<u>we deduce the following particular cases from the above equations:</u>

(i) if (u_2=0), from eqn. 2 (φ =0), if the sphere of mass m_2 were at rest, it will move along the line of centers after impact.

(ii) if (u_2=0)and $m_1 = e m_2$, from eqn. 2 (φ =0), and then from eqn. 5 (θ =0), so that if a sphere of mass m_1 impinges obliquely on a sphere of mass m_2 at rest, the direction of motion of the spheres after impact will be at right angles if $m_1 = e m_2$. This evidently hold true when the spheres are equal and perfectly elastic. (u_2=0), ($e = 1$) and ($m_1 = m_2$),

(iii) if $m_1 = m_2$,($e = 1$) then $v_1 \cos\theta = u_2 \cos\beta$ and $v_2 \cos\varphi = u_1 \cos\alpha$. i.e. if two equal and perfectly elastic spheres impinge they interchange their velocities in the direction of their line of centers.

Also by this case, we get $\tan\theta \tan\varphi = \tan\alpha \tan\beta$. If follows that, if two equal and perfectly elastic spheres impinge at right angles, their directions after impact will still be at right angles.

- ***<u>Losses in kinetic energy</u>***

In general there is always a loss of kinetic energy whenever two bodies impinge.

Loss in K.E = $\left(\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2\right) - \left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right)$

$$= \begin{array}{l} \left(\frac{1}{2}m_1\left(u_1^2 \cos^2\alpha + u_1^2 \sin^2\alpha\right) + \frac{1}{2}m_2\left(u_2^2 \cos^2\beta + u_2^2 \sin^2\beta\right)\right) \\ -\left(\frac{1}{2}m_1\left(v_1^2 \cos^2\theta + v_1^2 \sin^2\theta\right) + \frac{1}{2}m_2\left(v_2^2 \cos^2\varphi + v_2^2 \sin^2\varphi\right)\right) \end{array}$$

but the components of the velocities at right angles to the line of centers are unaltered and thus, we have

$v_1 \sin\theta = u_1 \sin\alpha$ and $v_2 \sin\varphi = u_2 \sin\beta$

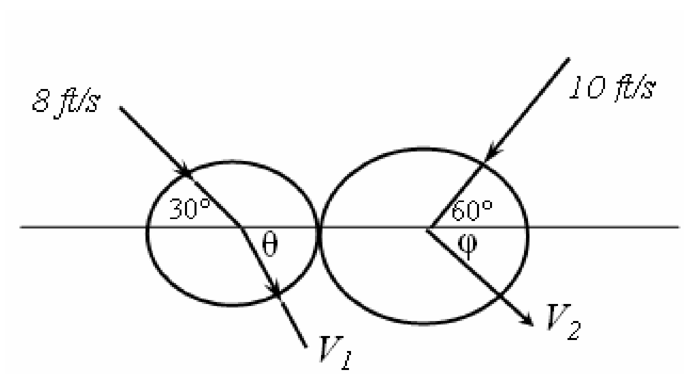
hence the loss in K.E= $\left(\frac{1}{2}m_1\left(u_1^2\cos^2\alpha\right)+\frac{1}{2}m_2\left(u_2^2\cos^2\beta\right)\right)-\left(\frac{1}{2}m_1\left(v_1^2\cos^2\theta\right)+\frac{1}{2}m_2\left(v_2^2\cos^2\varphi\right)\right)$

it is follows that it is only velocity components in the line of centers that can affect a change in the kinetic energy of the spheres.

Example (3)

A smooth sphere of mass **5** lb moving with a velocity **8** ft/s from a direction **30°** collides with a sphere of **3** lb moving with a velocity **10** ft/s determine the subsequent motion of the two spheres take (**e=0.6**).

Solution



since the spheres are smooth, the velocities perpendicular to the line of impact are equals

$v_1 \sin\theta = 8\sin 30^0$ ……………………..……………………..(1)

$v_2 \sin\varphi = 10\sin 60^0$ …..………………………………………(2)

since the momentum after impact along the line of centers = momentum before impact

$5v_1\cos\theta + 3v_2\cos\varphi = 5x8\cos 30^0 - 3x10\cos 60^0$ ………………(3)

by Newton's experimental law for relative velocities

$v_1\cos\theta - v_2\cos\varphi = -\frac{3}{5}\left(8\cos 30^0 + 10\cos 60^0\right)$ …………………(4)

from (3) and (4) we have

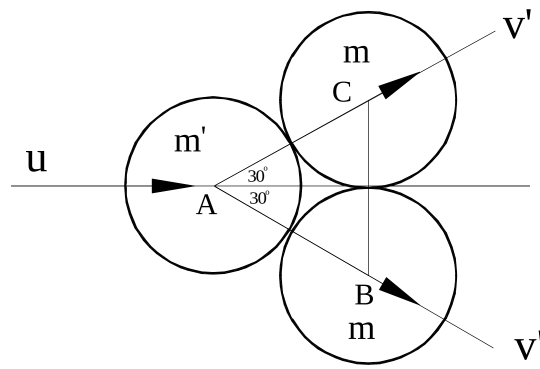
$3v_2 \cos\theta + 5v_2 \cos\varphi = 32\sqrt{3}$ or $v_2 \cos\varphi = 4\sqrt{3}$ ………………(5)

from (2) and (5), we have $v_2 = 11$ ft/sec and $\varphi = 51^0 20^{\backslash\backslash}$

Example (4)

Two equal balls (**m**) are lying in contact on a smooth table, a third ball (**m'**) moving with velocity (**u**) impinges symmetrically on them, prove that this ball is reduced to rest by the impact if **2m' = 3e m**.

Solution



Before impact, let **u** be the velocity of **m'**, which is reduced to rest after the impact

The spheres being equal in size and the common tangent makes equal angles each 30^0

By symmetry the velocities of each ball before impact be equal, say equal to v'.

The total momentum of spheres along the common tangent line will remain constant.

$$mv' \cos 30^0 + mv' \cos 30^0 = m'u \quad \text{or} \quad \sqrt{3}mv' = m'u \quad \text{……….} \quad (1)$$

by Newton's experimental law for relative velocities along AB or BC

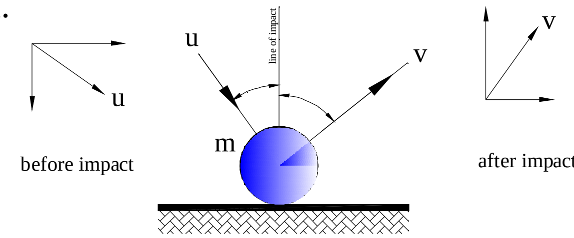
$$v' - 0 = -e\left(0 - u\cos 30^0\right) \quad \text{or} \quad v' = \frac{\sqrt{3}}{2}eu \quad \text{……………..} \quad (2)$$

from (1) and (2), we get

2m' = 3 e m.

6.5.3) Impact against a fixed plane.

Suppose a smooth sphere of mass m, moving with velocity u, strikes a smooth fixed plane in a direction making an angle α with the normal to the plane, and that it rebounds with velocity v making an angle θ with the normal. Since the plane is smooth, the component of the velocity along the plane must remain unaltered.



$v \sin\theta = u \sin\alpha$ ………………..(1)

The plane being fixed, its velocity is taken as zero.

By Newton's experimental law for relative velocity along the common normal (line of impact), we have

$v \cos\theta - 0 = -e(-u\cos\alpha - 0) \quad \Rightarrow \quad \therefore v\cos\theta = eu\cos\alpha$ ……………(2)

squaring and adding (1) by (2), we get $v^2 = u^2\left(\sin^2\alpha + e^2\cos^2\alpha\right)$ dividing (2) by (1) we have

$\cot\theta = e\cot\alpha$ …………………..(3)

These equations gives the velocity and direction of motion of the sphere after impact.

The following particular cases from the above equations:

(i) if $\alpha = 0$ then $\theta = 0$, therefore $v = eu$. i.e. when the impact is direct, the direction of motion of the sphere is reversed after impact and its velocity is reduced in the ration $e:1$.

(ii) if $e = 1$, $\theta = \alpha$ and then $v = u$. i.e. when the impact is perfectly elastic, the angle of reflection is equal to the angle of incidence, and the velocity remains unchanged in magnitude.

(iii) if $e = 0$, $\theta = 90^0$ and then $v = u\sin\alpha$. i.e. when the plane is perfectly inelastic, the sphere simply slides along the plane, its velocity parallel to the plane.

- Loss in K.E = $\frac{1}{2}mu^2 - \frac{1}{2}mv^2$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2\left(sin^2 a + e^2 cos^2 a\right) = \frac{1}{2}mu^2\left(1 - e^2\right)cos^2 a$$

- Impulse = change of momentum perpendicular to the plane

$$= mu\,cos\,a - \left(-mv\,cos\,q\right) = m\left(1+e\right)u\,cos\,a$$

<u>Example (5)</u>

A ball weighing one pound and moving with a velocity **8** ft/s impinges on a smooth fixed plane in a direction making **60°** with the plane, find its velocity and direction of motion after impact if the coefficient of restitution is **0.5** . Find also the loss of kinetic energy and the impulse of the plane due to the impact.

<u>Solution</u>

$$\left.\begin{array}{l} 8\cos 60° = v\cos f \\ \frac{1}{2}\left(8\sin 60°\right) = v\sin f \end{array}\right\}\text{by dividing } f = 40°\ \ 53' \quad and \quad v = 5.3\,ft/s$$

loss in K.E = $\frac{1}{2}m\left(u^2 - v^2\right) = \frac{1}{2}\left(\frac{1}{32}\right)\left(8^2 - 5.3^2\right) = \frac{9}{16}$

Impulse = $m\left(1+e\right)u\,cos\,a = \left(1\right)\left(1 + \frac{1}{2}\right)8\,cos\,30^0 = 10.4$ ft/sec

Example (6)

A particle falls from a height **h** in time **t** up on a fixed horizontal plane it rebounds and reaches the maximum height **h'** in time **t'**, show that **(t' = et)** and **(h' = e² h)** also prove that the whole distance (up and down) described by the particle before it has finished rebounding is **h[(1+e²)/(1-e²)]** and the time that elapses is **[(1+e)/(1-e)]**$\sqrt{\frac{2h}{g}}$ **.**

Solution

The time taken and velocity **u** acquired on reaching the horizontal plane are given by

$u^2 = 2gh$, and $u = gt$ or $t = \sqrt{\frac{2h}{g}}$ …………………………..(1)

the velocity of rebound being **eu,** the time t' and the max height h' attained by the particle to reach the highest point where its velocity is reduced to zero, are given by

$eu = gt'$, and $e^2u^2 = 2gh'$ or $t' = et$ …………………………(2)

it will reach the horizontal plane again with the same velocity **eu** in the same time **t=et'** after moving the same distance **h'=e²h**

thus the first rebound and the second rebound takes place, the particle takes **2et** and describes the total distance up and down equal to **2 e²h**.

the same process will be repeated subsequently till the particle comes to rest, thus the whole time taken by the particle during the motion = **t+2et+2e²t+……= t+2t(e+e²+…)**

= t+(2et/1-e) = t[1+(2e/1-e)]= (1+e/1-e) t = (1+e/1-e) $\sqrt{\frac{2h}{g}}$

and the total distance described by it **= h+2e²h+2e⁴h=h+(2e²h/1-e²)=(1+e²/1-e²) h**

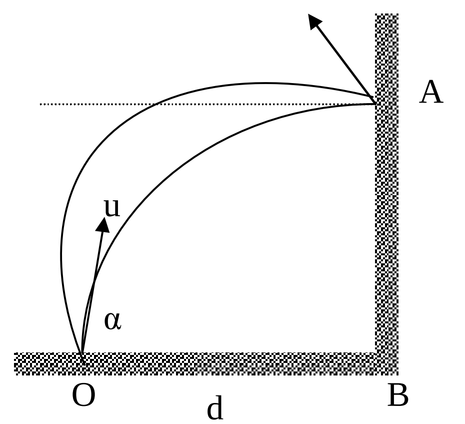
6.5.4) A Projectile impinging on a fixed plane.

A particle projected with a certain velocity in a certain direction, other than the horizontal and vertical may strike a horizontal, a vertical or an inclined plane, and may continue its motion after rebounding from the plane.

Example (7)

A ball is projected with velocity **u** at an elevation **α** from a point at a distance (**d**) from a smooth vertical wall in a plane perpendicular to it after rebounding from the wall it returns to the point of projection prove : **$u^2 \sin 2\alpha = gd\ (1 + 1/e)$**

Solution



Let the time taken by the particle to move from O to A

$$t_1 = \frac{d}{u \cos a}$$

and the time taken to return from A to O

$$t_2 = \frac{d}{eu \cos a}$$

hence the total time of the motion is $t = t_1 + t_2$

$$t = (t_1 + t_2) = \frac{d}{u \cos a}\left(1 + \frac{1}{e}\right) \text{ .... (1)}$$

since the vertical displacement of the ball is zero $\therefore\ y = 0$

$$0 = u \sin a\,(t_1 + t_2) - \tfrac{1}{2} g (t_1 + t_2)^2 \;\Rightarrow\; (t_1 + t_2) = \frac{2u \sin a}{g} \text{ .... (2)}$$

by equating (1) and (2) we have $\frac{d}{u \cos a}\left(1 + \frac{1}{e}\right) = \frac{2u \cos a}{g}$

$$gd\left(1 + \frac{1}{e}\right) = 2u^2 \sin a \cos a$$

$$\therefore gd\left(1+\frac{1}{e}\right)=u^2 \sin 2a$$

PROBLEMS

1- A ball impinges directly on a similar ball at **rest**. The first ball reduced to rest by the impact. Find the coefficient of restitution if the half of the initial kinetic energy is lost by impact.

2- A ball impinges directly on a other ball at **rest**. Prove that if the coefficient of restitution be equal to the ratio of their masses, the balls will leave in directions at right angles to each other.

3- A ball of mass **200-g** impinges on another equal ball at rest with velocity **1.5 m/s**. determine their final velocities just after impact if the coefficient of restitution is **e=0.85**.

4- A ball of mass **M** moving with a velocity **V** impinges with another of mass **m** at **rest**. Both are perfectly elastic (**e=1**) and the ball m is driven in a direction making an angle **θ** with the line of impact, show that its velocity is $\left(\frac{2M}{m+M}\right)V\cos q$.

5- A smooth ball impinges on another smooth equal ball at **rest** in a direction that makes an angle **α** with the line of centers at the moment of impact. Prove that if **D** be the angle through which the direction of the impinging ball is deviated then

$$\tan D=\frac{(1+e)\tan a}{1-e+2\tan^2 a}$$

6- A ball weighing **10-lb** and moving with a velocity **30** ft/s impinges on a smooth fixed plane in a direction making **60°** with the plane, find its velocity and direction of motion after impact if the coefficient of restitution is **2/3**. Find also the loss of kinetic energy and the impulse of the plane due to the impact.

7- A ball projected from a point O hits a vertical wall, rebounds and passes through the point **16** ft above O two seconds after projection. If the distance of the wall from O is **30** $\sqrt{3}$ ft and **e = 3/5**; find the magnitude and the direction of the initial velocity of the ball.